



UNIVERSIDAD POLITÉCNICA DE CARTAGENA

Departamento de Economía

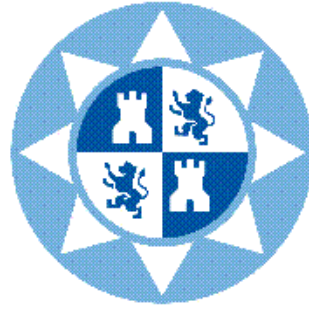
Área de Fundamentos del Análisis Económico

¿POR QUÉ LA VIRTUD ESTÁ EN EL PUNTO MEDIO?

PROBLEMAS DE RACIONAMIENTO CON PRINCIPIOS DE  
EQUIDAD LEGÍTIMOS.

José Manuel Jiménez Gómez

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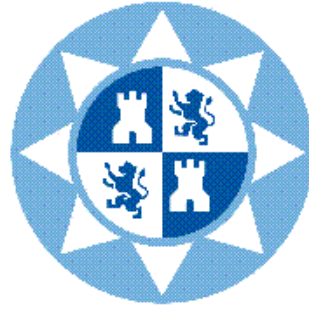
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Directores:

M<sup>a</sup> Carmen Marco Gil

Josep E. Peris Ferrando



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Área de Fundamentos del Análisis Económico

WHY DOES VIRTUE LIE IN THE MIDDLE GROUND?  
RATIONING PROBLEMS WITH COMMONLY ACCEPTED  
EQUITY PRINCIPLES.

José Manuel Jiménez Gómez

2010

Advisors:

M<sup>a</sup> Carmen Marco Gil

Josep E. Peris Ferrando

*A nuestro bebé, cuya existencia alberga la virtud.*

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## RESUMEN

*¿Cómo deberían repartirse los recursos escasos entre sus demandantes?*

*¿Debería cada individuo tener garantizada unas determinadas ganancias?*

Un problema de racionamiento aparece cada vez que la cantidad disponible de un determinado bien es insuficiente para satisfacer los “derechos objetivos” que, sobre el mismo, posee un grupo de agentes. El ejemplo más característico es el de la quiebra de una empresa, de aquí la denominación de este tipo de problemas en la literatura económica como “*Problemas de Bancarrota*” (*Bankruptcy Problems*). Sin embargo, la descripción previa es aplicable a muchas y muy diferentes situaciones reales, entre las que podemos mencionar: el reparto de una herencia que no alcanza el valor de los compromisos que sobre la misma se establecieron; la distribución de un producto, en un contexto de precios fijos, para el que la demanda excede a la oferta; o la recaudación de una cantidad fija de impuestos en una determinada comunidad.

Este tipo de problemas se ha descrito formalmente por un vector  $(E, c) \in \mathbb{R}_{++} \times \mathbb{R}_+^n$ , tal que  $E < \sum_{i \in N} c_i$ , donde  $E$  es la cantidad disponible del bien que tiene que distribuirse entre los agentes del conjunto  $N = \{1, \dots, i, \dots, n\}$  y cada coordenada  $c_i$  es interpretada como la cantidad que el agente  $i$ -ésimo demanda del bien, reflejando la condición  $E < \sum_{i \in N} c_i$  la incompatibilidad de los derechos individuales.

En este modelo, una solución denominada “*Regla de Bancarrota*” (*Bankruptcy rule*), debe entenderse como la aplicación de un proceso que proporciona una distribución del bien teniendo en cuenta las diferencias existentes entre los derechos de los agentes. A este respecto, la Teoría Económica se ha centrado en el estudio de

las diferentes soluciones desde tres enfoques diferentes, analizados en mayor detalle más adelante.

El objetivo principal del presente trabajo es el análisis exhaustivo, en el contexto de los problemas de racionamiento, de dos importantes preocupaciones generales: la discrepancia sobre las justificaciones morales de lo que se entiende por reparto justo, y la exploración sobre el establecimiento de medidas encaminadas a garantizar determinados derechos individuales.

En relación a la búsqueda de una solución cuando se distribuyen recursos escasos, la sociedad siempre ha tenido la preocupación de proponer asignaciones justas, como queda señalado por Lerner [27], entre otros: "*Las personas parecen tener el fuerte deseo de creer en un mundo justo.*" En este sentido, Schokkaert y Overlaet [46] destacan dos corrientes diferenciadas en la abundante literatura sobre distribución justa: filósofos-economistas y psicólogos-sociólogos. La primera, que es una línea más teórica, trata de recoger una interpretación aceptable de lo que entendemos por distribuciones justas mediante la definición formal de modelos. La segunda, que es una línea más informal y descriptiva, trata de explicar la manera de percibir la justicia por las personas, y su comportamiento cuando se enfrentan a problemas de distribución. Sin embargo, la *Justicia* difícilmente nos conduce a un único punto de vista: el mismo problema de distribución analizado por dos sociedades diferentes nos conducirá, casi con toda seguridad, al uso de diferentes reglas de reparto. (Moulin [35], Young [65], entre otros). En nuestro contexto, esta discrepancia aparece de manera natural, ya que cualquier distribución en los problemas de bancarrota puede ser observada desde dos puntos de vista: ganancias o pérdidas. Por lo tanto, tenemos dos puntos focales, que dependen de si nos preocupan las ganancias que recibimos o la cantidad no satisfecha de nuestra demanda. Asimismo, podemos encontrar muchos contextos diferentes donde el desacuerdo sobre la asignación propuesta queda recogido mediante dos puntos focales. Por ejemplo, en el contexto de los problemas

de reparto de excedente (surplus sharing problems), Moulin [35] argumenta que las reglas de reparto proporcional e igualitaria son las dos soluciones focales a estos problemas y, de hecho, son las únicas que satisfacen un *conjunto razonable de propiedades*. Del mismo modo, en la teoría de la justicia y asignaciones libres de envidia (envy-free allocations), Varian [59] ya propone dos distribuciones focales: justa-renta y justa-riqueza.

La pregunta, por tanto, es: *¿Cómo puede la gente alcanzar un acuerdo?* En palabras de Roemer [44], más información ayuda a que los acuerdos se alcancen: *"Las cosas se vuelven más interesantes una vez que dejamos el contexto restringido de la negociación del bienestar (welfarist bargaining) e incluimos información sobre recursos, preferencias, necesidades, habilidades, etcétera."*

Trasladando estas ideas a los problemas de bancarrota, el presente trabajo define un problema de racionamiento extendido, el *Problema de Bancarrota Lorenz-Bifocal (Lorenz-Bifocal Bankruptcy Problem)*, y realiza un profundo análisis sobre las consecuencias producidas por un enriquecimiento del modelo clásico de racionamiento con un tercer elemento,  $P$ , llamado conjunto de *Principios Legítimos (Legitimate Principles set)*, el cual recoge principios éticos básicos comúnmente aceptados por una sociedad para resolver una familia específica de problemas de este tipo. La aparición de desacuerdo también es estudiada con el establecimiento de dos reglas focales, digamos  $f$  y  $g$ , considerando tanto el hecho de que los problemas de racionamiento pueden analizarse desde ganancias o pérdidas, como que el objetivo general deseado por la sociedad es tratar a todo el mundo lo más igualitariamente posible. Esta idea queda recogida a través del *criterio de Lorenz* (Lorenz [28]), siguiendo la línea de trabajo de Arin [4] y Dutta y Ray [17], entre otros. Por lo tanto, nuestro problema se describe a través de un vector formado por un problema

de bancarrota y las dos reglas focales, que dependen del conjunto de *Principios Legítimos* acordado por una sociedad, esto es,  $((E, c), f(P), g(P))$ .

Con respecto al establecimiento de garantías, la preocupación de asegurar unos derechos individuales mínimos ha figurado de manera destacada en un gran número de contextos. En este sentido, un tema clásico que merece la pena destacar, ya que ha acaparado la mayor parte de la atención tanto de la agenda política como en la literatura político-social durante las dos últimas décadas, es la *Renta Básica Universal* (*Universal Basic Income*) (ver, por ejemplo, Tobin et al. [57]). Esta propuesta consiste en el pago por parte de la administración de una prestación monetaria a todos los ciudadanos, a la que tendrán derecho por la simple razón de existir como miembro pleno de una comunidad política, sin tener en cuenta la renta, vida laboral, capacidades laborales, o la composición de su familia (véase Noguera [41]). La progresión de esta idea en los círculos académicos durante los años recientes, es realmente notable, tanto en términos sociales como políticos. En este sentido, la *Red Mundial sobre la Renta Básica* (*Basic Income Earth Network*) ha propiciado el debate y la investigación sobre la *Renta Básica Universal* tanto en el Parlamento Europeo como en sus Comisiones y, en el caso de Brasil, ha sido aprobada una ley encaminada a implementar la *Renta Básica Universal* de manera indefinida. Otro contexto donde la idea de garantía aparece, entre otras situaciones, es el establecimiento de un salario mínimo en el mercado laboral o, más recientemente, el debate en Senado estadounidense acerca de asegurar una cobertura sanitaria mínima universal. Por otra parte, y llevado a nuestro contexto, notemos que, muchos autores, desde O'Neill [42] hasta Dominguez [15], han estudiado las garantías desde el punto de vista de los problemas de racionamiento.

Teniendo en cuenta estas dos prominentes preocupaciones generales, con el problema de racionamiento extendido,  $((E, c), f(P), g(P))$ , nosotros analizamos

las repercusiones producidas por la introducción de toda esta información adicional en el problema clásico de bancarrota, de acuerdo con cada uno de los enfoques metodológicos utilizados para abordar estos problemas. Consecuentemente, este estudio está estructurado en tres partes: Axiomático, Estratégico y Juegos Cooperativos.

- La Aproximación Axiomática, iniciada por Young [63, 64], trata de identificar reglas de reparto con conjuntos de propiedades estructurales y ha dado lugar a una amplia literatura, brillantemente sintetizada por Thomson [52] y Moulin [37]. En esta línea, que sigue en continuo devenir, se enmarca el reciente análisis del establecimiento de diferentes niveles de garantía en los repartos (veáse Herrero y Villar [23, 24], Moulin [37], Moreno-Ternero y Villar [33], Chun [10], Dominguez y Thomson [16] y Dominguez [15]). Asimismo, mediante la aplicación recursiva de las mismas, se definen nuevas soluciones. En este contexto pretendemos analizar las implicaciones, sobre el concepto de garantías, de establecer, desde el punto de partida, un conjunto de *Principios Legítimos* exigibles a las soluciones. Entonces, nosotros consideramos que cada agente debería recibir, al menos, la menor de las cantidades recomendadas por todas estas soluciones.
- El Enfoque Estratégico de los problemas de racionamiento considera diferentes procedimientos para solucionar las diferencias que los agentes pueden tener a la hora decidir cómo distribuir los recursos. Cada procedimiento induce un juego en el que los agentes proponen estratégicamente una regla de reparto, y la solución del juego consiste en la búsqueda de los *Equilibrios de Nash* del mismo (Nash [40]). Esta línea tiene su origen en las investigaciones de van Damme [58], Marco, Peris y Subiza [31] y Naeve-Steinweg [39], que han sido trasladadas a problemas de racionamiento por Chun [8] y Herrero [21]. El

análisis de este tipo de juegos con la restricción de que los agentes únicamente puedan proponer reglas de reparto que satisfagan el conjunto de *Principios Legítimos*, será nuestro objetivo en este contexto.

- El Enfoque de Juegos Cooperativos utiliza la teoría de juegos como una herramienta para encontrar soluciones a los problemas de racionamiento. Específicamente, asocia a cada problema un juego y define reglas de reparto mediante la aplicación de soluciones propuestas inicialmente para juegos, tanto cooperativos con utilidad transferible, línea iniciada por O'Neill [42], como juegos de negociación, conexión analizada por Dagan y Volij [14]. Esta metodología también ha sido explorada, entre otros, por Aumann y Maschler [5], Curiel, Maschler y Tijs [13], Dutta y Ray [17] y, Alcalde, Marco y Silva [1, 2]. En este nuevo contexto, en el que incorporamos un conjunto de *Principios Legítimos* en la definición del problema de racionamiento, nosotros investigamos la posibilidad de definir y analizar el juego natural TU cooperativo asociado, siguiendo la línea de O'Neill [42].

Nuestros resultados están organizados como sigue. En primer lugar, presentamos la notación y los modelos necesarios para una buena comprensión de la presente tesis. El Capítulo 1 introduce nuestro nuevo problema de racionamiento extendido, llamado *Problemas de Bancarrota Lorenz-Bifocales (Lorenz-Bifocal Bankruptcy Problems)*. A continuación, los siguientes dos capítulos recogen el análisis axiomático sobre este tipo de problemas. En particular, el Capítulo 2 propone un nuevo método para acotar ganancias, siguiendo la línea iniciada por O'Neill [42]. Empezando por el hecho de que una sociedad establece su propio conjunto de "*Principios de Equidad Comúnmente Aceptados*", nuestra propuesta asegura que todos los agentes reciban al menos la cantidad mínima de ganancias de acuerdo con todas las reglas *Admisibles* por tal sociedad. Debido a que esta nueva acotación inferior no agota, en general,

todos los recursos, entonces analizamos su aplicación recursiva, y recuperamos reglas clásicas de bancarrota.

Siguiendo en el Enfoque Axiomático, el Capítulo 3 define el proceso *Lorenz Doble Recursivo* (*Lorenz Double Recursive*), el cual consiste en la imposición recursiva de la acotación inferior sobre ganancias y su acotación superior asociada, al mismo tiempo. También proporcionamos una manera natural de justificar la combinación convexa de reglas de bancarrota, recuperando la media de dos reglas que representan maneras opuestas y extremas de distribuir los recursos.

El Capítulo 4 analiza los *Problemas de Bancarrota Lorenz-Bifocales* desde un punto de vista estratégico, obteniendo nuevas bases para reglas de bancarrota clásicas mediante la utilización de los mecanismos introducidos por Chun [8] y Herrero [23]. Además, a través de la definición de un nuevo proceso, combinación de los dos anteriores, recuperamos, de nuevo, la media de dos reglas que representan maneras opuestas y extremas de distribuir los recursos.

El Capítulo 5 propone una manera natural de asociar un juego de utilidad transferible a nuestro nuevo problema de bancarrota extendido, denominado *Juego de Bancarrota Lorenz-Bifocal* (*Lorenz-Bifocal Bankruptcy Game*). En particular, establecemos la condición necesaria que tienen que cumplir las soluciones para estar en el *Core* de este tipo de juegos, y, aunque estos juegos, en general, no son convexos, obtenemos que el *valor de Shapley* (*Shapley value*) es una selección del *Core* y que también coincide con el *Nucleolus*. Además, demostramos que la recomendación hecha por estos dos conceptos de solución es, otra vez, la media de dos reglas extremas.

Finalmente, el Capítulo 6 resume nuestras conclusiones generales e introduce algunos trabajos en progreso y futuras líneas de investigación.

A modo de síntesis, en esta tesis, por una parte, asegurando una mínima cantidad de ganancias a cada agente, proporcionamos nuevas bases, desde los Enfoques Axiomático y Estratégico, para reglas clásicas de bancarrota: *Igual Ganancia Restringida (Constrained Equal Awards)*, *Igual Pérdida Restringida (Constrained Equal Losses)*, *Piniles'* y su dual, e *Iguaitaria Restringida (Constrained Egalitarian)* y su dual. Del mismo modo, explicamos cómo la combinación de los acuerdos sociales sobre principios y el proceso recursivo puede no ser plausible.

Por otra parte, usando estos dos enfoques e imponiendo al mismo tiempo tanto una acotación inferior en ganancias como una superior, mostramos que podemos sostener de una manera razonable la propuesta de acuerdos "*intermedios*" en problemas de bancarrota, de entre todas aquellas soluciones que son consideradas *Admisibles* por una determinada sociedad, esto es, aquellas que satisfagan el conjunto de *Principios Legítimos* establecidos por y para la sociedad, ya que recuperamos la media entre dos asignaciones destacadas. Además, mediante la utilización del Enfoque de Juegos Cooperativos, también encontramos sólidas bases para seleccionar la media de dos puntos de vista focales de entre todos aquellos compromisos intermedios, proporcionando un soporte muy simple, actual y comúnmente observado en las decisiones colectivas.

Considerando todo lo anterior, pensamos que esta tesis arroja luz sobre la cuestión propuesta en el título de la misma:

*¿Por qué la virtud está en el punto medio?*

## ABSTRACT - INTRODUCTION

*How should the scarce resources be rationed among claimants?*

*Should each individual have a guaranteed level of awards?*

A rationing problem appears each time that the available quantity of a particular good is insufficient to satisfy the "objective rights" that a group of agents has on it. The most characteristic example is the bankruptcy of a company, hence the designation of such problems in the economic literature as *Bankruptcy Problems*. However, the previous description is applicable to many different real situations, among which we mention: the division of an inheritance which does not reach the value of the commitments that were established on it; the distribution of a product, in a context of fixed prices, for which demand exceeds supply; or the collection of a fixed amount of taxes in a given community.

This type of problem is formally described by a vector  $(E, c) \in \mathbb{R}_{++} \times \mathbb{R}_+^n$ , such that  $E < \sum_{i \in N} c_i$ , where  $E$  is the available quantity of the good to be distributed among agents in the set  $N = \{1, \dots, i, \dots, n\}$  and each coordinate  $c_i$  is interpreted as the demand for good by the  $i$ -th agent, meaning the condition  $E < \sum_{i \in N} c_i$  reflects the incompatibility of all rights.

In this model, a solution called *Bankruptcy Rule* involves applying a procedure for each bankruptcy problem that ensures a distribution of the good that takes the differences among agents' claims into account. In this regard, Economic Theory has focused on the study of different solutions from three different approaches, discussed in further detail below.

The main purpose of this dissertation is to produce an in-depth analysis, in the context of rationing problems, of two important general concerns: discrepancy over the moral intuitions of a fair distribution, and the prospect of establishing measures to guarantee certain individuals' rights.

In terms of finding a solution, society has always taken care to propose a fair allocation when distributing resources, as pointed out by Lerner [27], among others: "*People seem to have a strong desire to believe in a just world.*" In this regard, Schokkaert and Overlaet [46] highlight two streams in the vast literature on distributive justice: one of philosophers-economists, and one of psychologists-sociologists. The former, more theoretical line, defines formal models trying to gather an acceptable interpretation of fair distributions. The latter, more informal and descriptive line, tries to explain the way in which people perceives fairness, and their behavior when facing distribution problems. However, *Fairness* hardly leads to a single viewpoint: the same distribution problem faced by two different societies will almost certainly lead to the use of different distributional rules (Moulin [35], Young [65], among others). In our framework, this discrepancy arises in a natural way, as any distribution in bankruptcy problems can be observed by focusing either on gains or on losses. We thus have at least two focal viewpoints, depending on whether we worry about the awards we receive or the amount of our demand that is not satisfied. We can also find many contexts where discrepancy over the proposed allocation results in two focal points. For instance, in the context of the surplus sharing problem, Moulin [35] argues that the equal and proportional sharing rules are the two focal solutions of this problem and, indeed, the only ones satisfying a *reasonable set of properties*. Furthermore, in the theory of fairness and envy-free allocation, Varian [59] has already proposed two focal distributions: an income-fair and a wealth-fair allocations basing on abilities.

The question, therefore, is: *How can people reach an agreement?* In Roemer's [44] words, more information helps agreements to be reached: *"Things become more interesting once we leave the restricted welfarist of bargaining framework and include information on resources, preferences, needs, skills and so on."*

Translating these ideas to bankruptcy problems, the current work defines an extended rationing problem, the *Lorenz-Bifocal Bankruptcy Problem*, and makes a thorough analysis of the consequences of enriching the classical model of rationing with a third element,  $P$ , called set of *Legitimate Principles*, comprising basic ethical principles commonly accepted by a society to resolve a specific family of problems of this kind. The appearance of discrepancy is also analyzed, with the establishment of two focal rules, say  $f$  and  $g$ , by considering both the fact that rationing problems can be analyzed from awards or losses, and that the general desirable social goal is to treat everybody as evenly as possible. This idea is captured by the *Lorenz criterion* (Lorenz [28]), in line with the work of Arin [4] and Dutta and Ray [17], among others. Thus, our problem is described by the vector formed by a bankruptcy problem and the two focal allocations, which depend on the set of *Legitimate Principles* accorded by a society, that is,  $((E, c), f(P), g(P))$ .

Regarding to establishing guarantees, the concern of ensuring minimum individual rights has been figured prominently in a high number of contexts. In this sense, a classical issue that is worth highlighting, as it has captured most of the attention in the social policy literature and the political agenda during the last two decades, is the *Universal Basic Income* (see for instance Tobin et al. [57]). This proposal involves the payment of a universal cash benefit to all citizens by the Administration, to which they would be entitled by the simple fact of being a full member of a community politics, regardless of income, employment history, availability for work, or the composition of her family (see for instance Noguera

[41]). The progression of this idea in academic circles in recent years has been truly remarkable, in both social and political terms. In this regard, the *Basic Income Earth Network* has led to debate and research into the *Universal Basic Income* in the European Parliament and Commission and, in the case of Brazil a law has been approved to implement the *Universal Basic Income* for an indefinite period. Another context where the idea of guarantee appears, among other situations, is the establishment of a minimum wage in the labour market or, more currently, the U.S. Senate's debate of ensuring universal minimum health coverage. On the other hand, note that in our context, many authors, from O'Neill [42] to Dominguez [15], have dealt with guarantees in rationing problems.

Taking into the account these two general and prominent concerns, with the extended rationing problem,  $((E, c), f(P), g(P))$ , we analyze the consequences of entering this additional information into the classical bankruptcy problem, according to each of the methodological approaches used to address these problems. Thus, this study is structured in three parts: Axiomatic, Strategic and Cooperative Games.

- The Axiomatic Approach, initiated by Young [63, 64], tries to identify bankruptcy rules with sets of structural properties and has led to a vast literature, brilliantly synthesized by Thomson [52] and Moulin [37]. In continuous evolution, this approach fits the recent analysis of establishing different levels of guarantees, when distributing the scarce resources (see Herrero and Villar [23, 24], Moulin [37], Moreno-Ternero and Villar [33], Chun [10], Dominguez and Thomson [16] and Dominguez [15]). Furthermore, by recursively applying these lower bounds, new solutions are defined along these lines. In this context we analyze the implications for the concept of guarantees (lower bounds) by establishing from the beginning a set of *Legitimate Principles* required for the solution allowed. We then naturally

consider that agents should receive at least the smallest amount recommended by all these solutions.

- Strategic analysis of rationing problems considers different procedures for solving the differences between agents when deciding how to distribute the resource. Each procedure induces a game in which players strategically propose a bankruptcy rule, and the solution of the game consists of finding the *Nash equilibria* (Nash [40]). This approach comes from research by van Damme [58], Marco, Peris and Subiza [31] and Naeve-Steinweg [39], transferred to rationing problems by Chun [8] and Herrero [21]. Our aim is to analyze these kind of games with the restriction that agents only can propose allocations that satisfy the set of *Legitimate Principles*.
- The Cooperative Game Approach uses game theory as a tool for finding solutions to bankruptcy problems. Specifically, it relates a game each rationing problem and defines distribution rules by applying solutions originally proposed for games, both cooperative with transferable utility, initiated by O'Neill [42], and bargaining games, as analyzed by Dagan and Volij [14]. This methodology has also been explored by Aumann and Maschler [5], Curiel, Maschler and Tijs [13], Dutta and Ray [17], and Alcalde, Marco and Silva [1, 2], among others. In our new context, in which we incorporate a set of *Legitimate Principles* into the definition of a rationing problem, we investigate the possibility of defining and analyzing the natural associated game, following the definitions of O'Neill [42] and Dagan and Volij [14].

Our results are organized as follows. Firstly, we present all the notation and the models needed for a good comprehension of the present dissertation. Chapter 1 introduces our new extended rationing problem, known as *Lorenz-Bifocal Bankruptcy*

*Problems*. Then, the following two chapters set out the axiomatic analysis of these problems. In particular, Chapter 2 proposes a new method for bounding awards following the line initiated by O'Neill [42]. Starting from the fact that a society establishes its own set of "*Commonly Accepted Equity Principles*", our proposal ensures all agents the minimum amount they can receive according to all the *Admissible* rules for such a society. Because this new lower bound will not exhaust the endowment in general, we then analyze its recursive application by retrieving old well known bankruptcy rules.

Following in this Axiomatic Approach, Chapter 3 defines the *Lorenz Double Recursive* procedure, which consists of the recursive imposition of both the lower bound on awards and its associated upper bound. We also provide a natural way of justifying the convex combination of bankruptcy rules, by means of their recursive application, retrieving the average of two extreme and opposite ways of distributing the endowment.

Chapter 4 analyses the *Lorenz-Bifocal Bankruptcy Problems* from an strategic viewpoint, and a new basis for old bankruptcy rules is obtained by using the mechanisms introduced by Chun [8] and Herrero [23]. Furthermore, by defining a new procedure, which is a combination of the previous ones, we retrieve the average of rules representing two extreme and opposite ways of distribute the resource.

Chapter 5 proposes a natural way of associating a TU game to our new extended bankruptcy problem, called the *Lorenz-Bifocal Bankruptcy Game*. Specifically, we state a necessary condition for solutions to be in the *Core* of these games; and, although these games are not generally convex, we find out that the *Shapley value* is a *Core* selection and also coincides with the *Nucleolus*. We also show that the recommendation made by these two solution concepts is again the average of the two extreme rules.

Finally, Chapter 6 summarizes our general conclusions and introduces some works in progress and future research lines.

On the one hand, therefore, by ensuring a minimal amount of awards to each claimant, we provide new basis from Axiomatic and Strategic viewpoints, for classical and well-known bankruptcy rules: the *Constrained Equal Awards*, the *Constrained Equal Losses*, *Finiles'* and its dual, and the *Constrained Egalitarian* and its dual. We also explain how the combination of social agreements on principles and recursive procedure may become implausible.

On the other hand, by using these approaches and by requiring both a lower and an upper bound on awards, we can reasonably sustain the proposal of "*intermediate*" agreements for bankruptcy problems, from among all the solutions that are considered *Admissible* for a given society, i.e., the ones satisfying the set of *Legitimate Principles* established by and for the society (we retrieve the average between two prominent allocations). Moreover, by using the Cooperative Game Approach, we also find solid grounds for selecting the average of the two focal viewpoints from among all the intermediate compromises, supporting very simple, current and commonly observed collective decisions.

Considering all the above, we think that the current dissertation sheds light onto the question posed in its title:

*Why does virtue lie in the middle ground?*



## PRELIMINARIES.

Next, we introduce all the concepts and definitions which are used in the following chapters. Firstly, we present rationing problems, highlighting some of their main solutions and paying special attention to the Axiomatic Approach: a method which consists of identifying bankruptcy rules with various combinations of appealing properties. Secondly, we introduce the main results on the establishment of guarantees when facing bankruptcy problems. Thirdly, we provide some definitions related to the *Lorenz criterion*, one of the most used and widely accepted egalitarian concepts. Finally, we present two alternative methods, based in Game Theory, that have been used to analyze bankruptcy situations: the Strategic and the Cooperative Game Approaches.

**0.1 Bankruptcy problems: Axiomatic Approach.**

The problem of allocating scarce resources among alternative uses is one of the elements that defines the nature of economic science. In this context, many different situations are fit, ranging from individual to social decision problems. For example, dividing our time between labour and leisure; how to spend our money; distributing the tax burden; the allocation of the budget estate among each ministry; rationing a natural resource, like water, among citizens, ... Particularly, our work focuses on analyzing those situations where a perfectly divisible good should be rationed among agents, since their aggregate demands exceeds its supply. Formally:

**Definition 1** A *bankruptcy problem* is a pair  $(E, c) \in \mathbb{R}_{++} \times \mathbb{R}_+^n$  such that

$$E < \sum_{i \in N} c_i.$$

Each agent  $i \in N$ ,  $N = \{1, \dots, i, \dots, n\}$ , has a claim  $c_i$  on the estate or endowment,  $E$ , which represents the quantity of a perfectly divisible good that should be distributed among the agents. Figure 1 shows this situation graphically. The black lines represent different level of the estate  $(E, E')$ , and the vector  $(c_1, c_2)$  shows the claims of the two agents, whose aggregate is bigger than the resources .

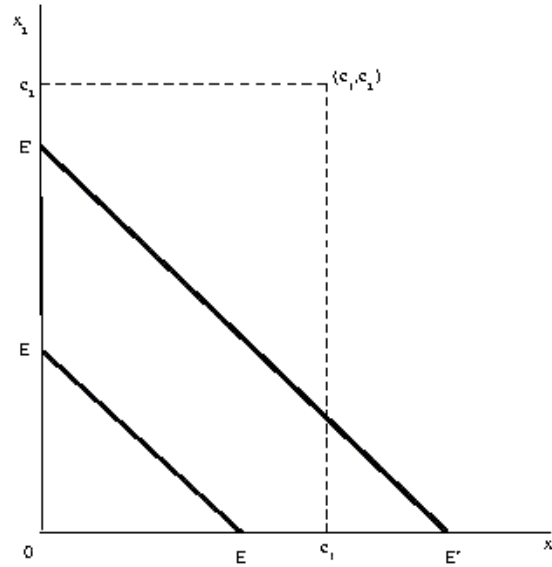


Figure 1: Bi-personal bankruptcy problems.

Hereinafter,  $\mathcal{B}$  denotes the set of all bankruptcy problems; given a bankruptcy problem  $(E, c) \in \mathcal{B}$ ,  $C$  denotes the sum of the agents' claims,  $C = \sum_{i \in N} c_i$ ; and  $\mathcal{B}_0$  the set of bankruptcy problems in which claims are increasingly ordered, that is, bankruptcy problems with  $c_i \leq c_j$  for  $i < j$ .

In this situation, the question is how to distribute a quantity of a perfectly divisible good among a group of agents whose aggregate demands, needs or rights, gathered by the claims vector, exceed the amount to be distributed. Its answer has been analyzed by different methods: the Axiomatic, the Strategic and the Cooperative Game Approaches. In this section we pay special attention to the former one.

From the axiomatic viewpoint, initiated by Young [63, 64], bankruptcy problems are treated from an external agent's perspective. Specifically, this "*arbitrator*" pursue the goal of deciding a "*fair*" sharing of the resource among the claimants, for which she requires that this distribution satisfies some appealing principles. So that, the Axiomatic Approach is a normative analysis bases on the search of properties, representing equity principles, to describe the family of solutions satisfying these requirements and, then, characterize each solution with the lowest number of principles (see, for instance, Thomson [52]).

Henceforth, next we define the concept of bankruptcy solution, named bankruptcy rule, and present some of the most important ones below. Thereupon, we introduce some appealing properties, particularly those which are playing an important role in the explanation of the current work.

**Definition 2** *A bankruptcy rule, or simply a **rule**, is a function,  $\varphi : \mathcal{B} \rightarrow \mathbb{R}_+^n$ , such that for each  $(E, c) \in \mathcal{B}$ ,*

$$(a) \sum_{i \in N} \varphi_i(E, c) = E \text{ (efficiency) and}$$

$$(b) 0 \leq \varphi_i(E, c) \leq c_i \text{ for each } i \in N \text{ (non-negativity and claim-boundedness).}$$

The *Proportional* rule makes awards proportional to the claims.

The **Proportional rule**,  $P$ , associates for each  $(E, c) \in \mathcal{B}$ , and each  $i \in N$ ,  $P_i(E, c) \equiv (E/C) c_i$ .

The *Constrained Equal Awards* rule (Maimonides, 12th century, among others) recommends equal awards to all claimants subject to no-one receiving more than her claim.

The **Constrained Equal Awards** rule,  $CEA$ , associates for each  $(E, c) \in \mathcal{B}$  and each  $i \in N$ ,  $CEA_i(E, c) \equiv \min \{c_i, \mu\}$ , where  $\mu$  is chosen so that  $\sum_{i \in N} \min \{c_i, \mu\} = E$ .

The *Constrained Equal Losses* rule, discussed by Maimonides (Aumann and Maschler [5]), chooses the awards vector at which losses from the claims vector are the same for all agents, subject to no-one receiving a negative amount.

The **Constrained Equal Losses** rule,  $CEL$ , associates for each  $(E, c) \in \mathcal{B}$  and each  $i \in N$ ,  $CEL_i(E, c) \equiv \max \{0, c_i - \mu\}$ , where  $\mu$  is chosen so that  $\sum_{i \in N} \max \{0, c_i - \mu\} = E$ .

*Piniles'* rule (Piniles [43]) provides, for each problem  $(E, c) \in \mathcal{B}$ , the awards that the *Constrained Equal Awards* rule recommends for  $(E, c/2)$ , when the endowment is less than the half-sum of the claims. Otherwise, each agent first receives her half-claim, then the *Constrained Equal Awards* rule is applied to the residual problem  $(E - C/2, c/2)$ .

**Piniles'** rule,  $Pin$ , associates for each  $(E, c) \in \mathcal{B}$  and each  $i \in N$ ,

$$Pin_i(E, c) \equiv \begin{cases} CEA_i(E, c/2) & \text{if } E \leq C/2 \\ c_i/2 + CEA_i(E - C/2, c/2) & \text{if } E \geq C/2 \end{cases}$$

The *Talmud* rule (Aumann and Maschler [5]), assigns the awards that the *Constrained Equal Awards* rule recommends for  $(E, c/2)$ , when the endowment is less than the half-sum of the claims. Otherwise, each agent first receives her half-claim, then the *Constrained Equal Losses* rule is applied to the residual problem  $(E - C/2, c/2)$ .

The **Talmud** rule,  $T$ , associates for each  $(E, c) \in B$ , and each  $i \in N$ ,

$$T_i(E, c) \equiv \begin{cases} CEA_i(E, c/2) & \text{if } E \leq C/2 \\ c_i/2 + CEL_i(E - C/2, c/2) & \text{if } E \geq C/2 \end{cases}$$

The *Constrained Egalitarian* rule, proposed by Chun et al. [11], is inspired by the *Uniform* rule (Sprumont [51]), a solution to the problem of fair division when the preferences are single-peaked. It makes the minimal adjustment in the formula of the *Uniform* rule, taking the half-claims as the peaks and guaranteeing that awards are ordered in the same way as claims are.

The **Constrained Egalitarian** rule,  $CE$ , associates for each  $(E, c) \in B$  and each  $i \in N$ ,

$$CE_i(E, c) \equiv \begin{cases} CEA_i(E, c/2) & \text{if } E \leq C/2 \\ \max\{c_i/2, \min\{c_i, \delta\}\} & \text{if } E \geq C/2 \end{cases}$$

where  $\delta$  is chosen so that  $\sum_{i \in N} CE_i(E, c) = E$ .

Next, Figures 2, 3, 4 and 5 show the path of awards for different levels of the resource, recommended by all the bankruptcy rules we have just introduced: the *Proportional* (red line), *Constrained Equal Awards* (blue line), the *Constrained Equal Losses* (green line), *Piniles'* (yellow), the *Talmud* (orange line) and the *Constrained Egalitarian* (grey line) rules.

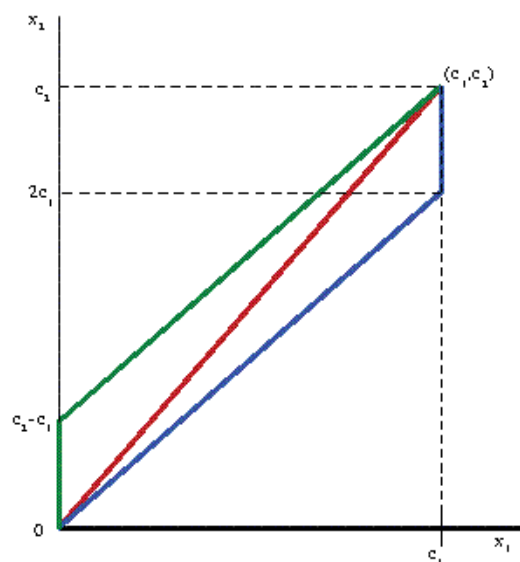


Figure 2: The *Constrained Equal Awards*, *Constrained Equal Losses* and *Proportional* rules.

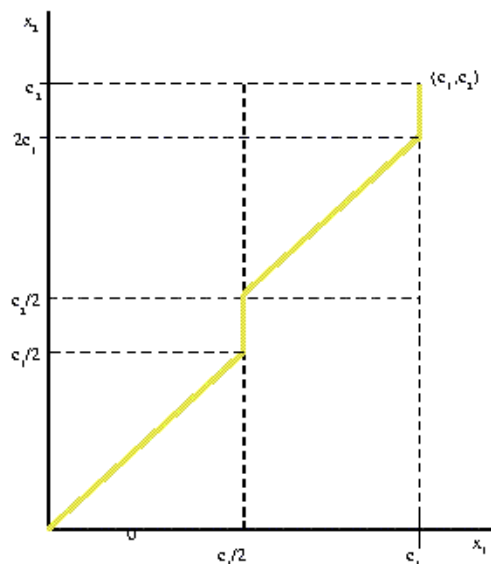


Figure 3: *Piniles'* rule.

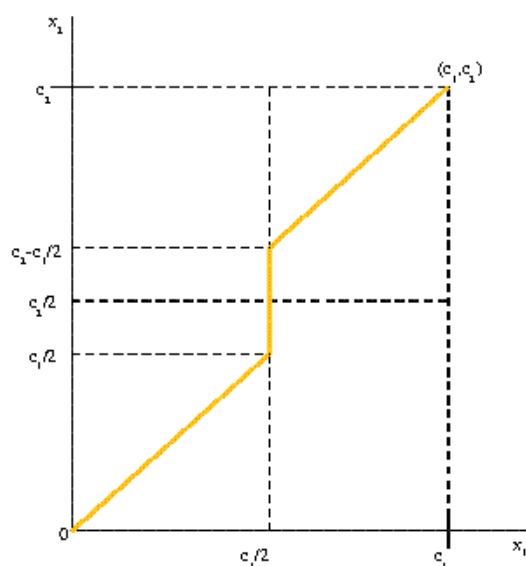


Figure 4: The *Talmud* rule.

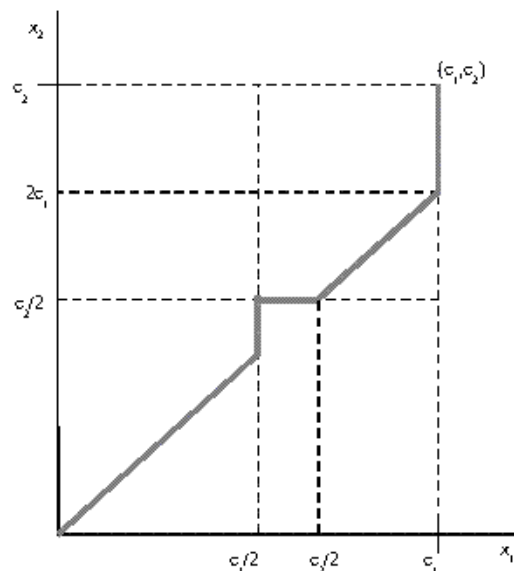


Figure 5: The *Constrained Egalitarian* rule.

At this point, it is worth noting that there are two natural viewpoints when facing bankruptcy problems. On the one hand, these situations could be analyzed taking into account the awards received by each agent, i.e., focusing on "*what is received*". Whereas, on the other hand, these problems can be treated by paying attention to the amount of the claims that are not satisfied, "*what is missing*", that is, the losses incurred by each agent when a rule  $\varphi$  is applied, denoted by  $l_i = c_i - \varphi_i(E, c)$ , for each  $i \in N$ . Then, the total loss to distribute,  $L$ , is the aggregate losses,  $L = \sum_{i \in N} l_i = C - E$ .

In this line, Aumann and Maschler [5] define the dual relation between rules as follows. Given a rule  $\varphi$ , its dual shares out losses in the same way that  $\varphi$  divides the endowment. Formally:

The **dual** of  $\varphi$ , denoted by  $\varphi^d$ , assigns for each  $(E, c) \in \mathcal{B}$  and each  $i \in N$ ,  $\varphi_i^d(E, c) = c_i - \varphi_i(L, c)$ .

It is straightforward to check that for each rule,  $\varphi$ , its dual is well defined, since given that  $(E, c) \in \mathcal{B}$ ,  $(L, c) \in \mathcal{B}$  and given that  $\varphi$  satisfies efficiency, non-negativity and claim-boundedness, the same will apply for  $\varphi^d$ .

Moreover, if a rule recommends the same allocation when dividing awards and losses, it is *Self-Dual*.

A rule  $\varphi$  is **Self-Dual**, if for each  $(E, c) \in \mathcal{B}$  and each  $i \in N$ ,  $\varphi_i(E, c) = c_i - \varphi_i(L, c)$ .

It can easily be checked that the *Proportional* and the *Talmud* rules are *Self-Dual*, while the *Constrained Equal Awards* and the *Constrained Equal Losses* rules are *dual* each other (Herrero [21]). Moreover, next we define the dual rules of the *Piniles'* and the *Constrained Egalitarian* ones.

*Dual of Piniles'* rule provides, for each problem  $(E, c) \in \mathcal{B}$ , the sharing that the *Constrained Equal Losses* rule recommends for  $(E, c/2)$ , when the endowment is less than the half-sum of the claims. Otherwise, each agent first receives her half-claim, then the *Constrained Equal Losses* rule is re-applied to the residual problem  $(E - C/2, c/2)$ .

**Dual of Piniles'** rule,  $DPin$ , associates for each  $(E, c) \in \mathcal{B}$  and each  $i \in N$ ,

$$DPin_i(E, c) = \begin{cases} CEL_i(E, c/2) & \text{if } E \leq C/2 \\ c_i/2 + CEL_i(E - C/2, c/2) & \text{if } E \geq C/2 \end{cases}$$

The *Dual Constrained Egalitarian* rule gives the half-claims a central role and makes the minimal adjustment in the formula of the *Dual Uniform* rule to guarantee that losses are ordered in the same way as claims are.

**Dual Constrained Egalitarian** rule,  $DCE$ , associates for each  $(E, c) \in \mathcal{B}$  and each  $i \in N$ ,

$$DCE_i(E, c) \equiv \begin{cases} c_i - \max \{c_i/2, \min \{c_i, \delta\}\} & \text{if } E \leq C/2 \\ c_i/2 + CEL_i(E - C/2, c/2) & \text{if } E \geq C/2 \end{cases}$$

where  $\delta$  is chosen such that  $\sum_{i \in N} DCE_i(E, c) = E$ .

Next, we will commenting on some appealing properties, particularly those which are playing an important role in the explanation of the current work.

*Order Preservation* (Aumann and Maschler [5]) requires respecting the ordering of the claims: if agent  $i$ 's claim is at least as large as agent  $j$ 's claim, he should receive and loss at least as much as agent  $j$ , does respectively.

**Order Preservation:** for each  $(E, c) \in \mathcal{B}$ , and each  $i, j \in N$ , such that  $c_i \geq c_j$ , then  $\varphi_i(E, c) \geq \varphi_j(E, c)$ , and  $c_i - \varphi_i(E, c) \geq c_j - \varphi_j(E, c)$ , that is  $l_i(E, c) \geq l_j(E, c)$ .

*Resource Monotonicity* (Curiel et al. [13], Young [64], among others) demands that if the endowment increases, then all individuals should get at least what they received initially.

**Resource Monotonicity:** for each  $(E, c) \in \mathcal{B}$  and each  $E' \in \mathbb{R}_+$  such that  $C > E' > E$ , then  $\varphi_i(E', c) \geq \varphi_i(E, c)$ , for each  $i \in N$ .

A *Super-Modular* rule (Dagan et al. [14]) allocates each additional dollar in an "order preserving" manner. In other words, when the endowment increases, agents with higher claims receive a greater part of the increment than those with lower claims.

**Super-Modularity:** for each  $(E, c) \in \mathcal{B}$ , all  $E' \in \mathbb{R}_+$  and each  $i, j \in N$  such that  $C > E' > E$  and  $c_i \geq c_j$ , then  $\varphi_i(E', c) - \varphi_i(E, c) \geq \varphi_j(E', c) - \varphi_j(E, c)$ .

*Midpoint Property* (Chun, Schummer and Thomson [11]) requires that if the estate is equal to the sum of the half-claims, then all agents should receive their half-claim.

**Midpoint Property:** for each  $(E, c) \in \mathcal{B}$  and each  $i \in N$ , if  $E = C/2$ , then  $\varphi_i(E, c) = c_i/2$ .

The dual relation defined between rules has been carried to the concept of property. In this sense, given two properties, we say that they are dual of each other if whenever a rule satisfies one of them, its dual satisfies the other.

Two properties,  $\mathcal{P}$  and  $\mathcal{P}'$ , are **dual** if whenever a rule,  $\varphi$ , satisfies  $\mathcal{P}$ , its dual,  $\varphi^d$ , satisfies  $\mathcal{P}'$ .

It is worth noting that all the principles we have introduced are invariant to the perspective from which the problem is thought, that is, they do not change when dividing "what is available" or "what is missing", so, they are *Self-Dual*. Formally:

A property,  $\mathcal{P}$ , is **Self-Dual** when it coincides with its dual.

## 0.2 Bounds on bankruptcy problems.

When analyzing bankruptcy problems from the axiomatic viewpoint many authors have proposed certain lower bounds on awards with the aim of ensuring a minimum amount of the resources to each agent. In fact, the solution for the "*Contested Garment Problem*" proposed in the Babylonian Talmud suggests that each agent should receive at least some part of the available amount when facing these kind of situations. This idea appears in the formal definition of a rule, requiring that no agent receives less than zero. Later, it has underlied the theoretical analysis of bankruptcy problems from its beginning (O'Neill, [42]) to present day (Dominguez, [15]). Next we chronologically present the main concepts that can be found in the literature around the idea of establishing lower bounds on awards.

- By requiring non-negativity,  $\varphi_i(E, c) \geq 0$  for each  $i \in N$ , the definition of a **bankruptcy rule** gives a lower bound on awards (see Definition 2).
- O'Neill [42] defines a lower bound on awards, called *Respect of Minimal Right*, which implies that each agent must receive at least, either the remaining endowment after the other claimants have been fully compensated, or nothing if this amount is negative.

A rule  $\varphi$  satisfies **Respect of Minimal Right** if for each  $(E, c)$  in  $\mathcal{B}$ , and for each  $i \in N$ ,

$$\varphi_i(E, c) \geq \max \left\{ 0, E - \sum_{j \in N \setminus \{i\}} c_j \right\}.$$

- Herrero and Villar [23, 24] introduce two properties, *Sustainability* and *Exemption*, that bound awards.

The former, *Sustainability* says that an agent should receive all her claim, whenever this is sustainable, defined as follows. A claim,  $c_i$ , is sustainable if when we truncate all the claims by this one, the problem becomes feasible, i.e.,  $\sum_{j \in N} \min \{c, c_j\} \leq E$ , where  $c = c_i$ .

A rule  $\varphi$  satisfies **Sustainability** if for each  $(E, c) \in \mathcal{B}$ , and each  $i \in N$ , if  $c_i$  is sustainable then,

$$\varphi_i(E, c) = c_i.$$

The second bound, *Exemption*, requires that when the amount to be divided is large enough with respect to agent's claims, only those agents with larger claims have to be rationed.

A rule  $\varphi$  satisfies **Exemption** if for each  $(E, c) \in \mathcal{B}$ , and each  $i \in N$ , if  $c_i \leq \frac{E}{n}$ ,

$$\varphi_i(E, c) = c_i.$$

- Moulin [37] defines a bound on awards, called *Lower Bound*, requiring that each agent receives at least a "fair" share of the resources except those who demand less than  $E/n$ , in which case her demand is met in full.

A rule  $\varphi$  satisfies **Lower Bound** if for each  $(E, c) \in \mathcal{B}$ , and each  $i \in N$ ,

$$\varphi_i(E, c) \geq \min \left\{ c_i, \frac{1}{n} E \right\}.$$

- Moreno-Tertero and Villar [33] present a weaker notion of Moulin's lower bound, called *Securement*, which establishes that each agent should receive at least  $1/n$  of her claim truncated by the amount to divide.

A rule  $\varphi$  satisfies **Securement** if for each  $(E, c) \in \mathcal{B}$ , and each  $i \in N$ ,

$$\varphi_i(E, c) \geq \frac{1}{n} \min \{c_i, E\}.$$

- Finally, Dominguez [15] proposes, under the name of *Min Lower Bound*, that each agent should receive at least  $1/n$  of the smallest claim truncated by the amount to divide.

A rule  $\varphi$  satisfies **Min Lower Bound** if for each  $(E, c) \in \mathcal{B}$ , and each  $i \in N$ ,

$$\varphi_i(E, c) \geq \frac{1}{n} \min \left\{ \min \{c_j\}_{j \in N}, E \right\}.$$

### 0.3 The Lorenz criterion.

Note that, as Sections 0.1 and 0.2 show, an important purpose when analyzing bankruptcy problems from the axiomatic viewpoint is to identify those rules fulfilling a set of requirements with which are pursued "fair" allocations. In this sense, when talking about "fairness", we should take into account that societies do not care only about the individual gains, but also about the social benefits. In Young's [65] words:

*"The sharing rules express a notion of equality in the division of jointly produced goods. By equitable I do not necessarily mean ethical or moral, but that which a given society considers to be appropriate to the need, status and contributions of various members".*

The equity criteria we apply in this work is the *Lorenz-criterion* which has been widely accepted when the general social goal is to treat everybody as evenly as possible (see for instance Arin [4], Dutta and Ray [17], Cowell [12] and Lambert [26]). In this regard, if a vector  $x$  *Lorenz dominates* a vector  $y$ , then  $x$  is less unequally distributed than  $y$ . Formally:

For each vector  $x \in \mathbb{R}_+^n$ , we denote by  $\Pi(x)$  the vector that results from  $x$  by permuting the coordinates in such a way that  $\Pi_1(x) \leq \Pi_2(x) \leq \dots \leq \Pi_n(x)$ .

Let  $x, y \in \mathbb{R}_+^n$ , we say that  $x$  **Lorenz dominates**  $y$ , denoted by  $x \succ_L y$ , if  $\Pi_1(x) \geq \Pi_1(y)$ ,  $\Pi_1(x) + \Pi_2(x) \geq \Pi_2(x) + \Pi_2(y)$ , and so on, with at least one strict inequality. Note that, given  $x, y \in \mathbb{R}_+^n$ , we do not impose on these vectors the condition  $\sum_{i \in N} x_i = \sum_{i \in N} y_i$  in order to apply the *Lorenz* domination criterion (see Arin [4]).

If the partial sums are equal for  $\Pi(x)$  and  $\Pi(y)$ , the two vectors,  $x$  and  $y$  are said to be **Lorenz equivalent**, denoted by  $x \sim_L y$ .

Moreover, a vector  $x \in \mathbb{R}_+^n$  is **Lorenz Maximal** if there is no other vector  $y \in \mathbb{R}_+^n$  such that,  $y \succ_L x$ . And, particularly, given a set  $S \subseteq \mathbb{R}_+^n$ , a vector  $x \in S$  is **Lorenz Maximal** in  $S$  if there is no other vector  $y \in S$  such that,  $y \succ_L x$ .

Finally, applying the *Lorenz-criterion* on the two focus, gains and losses, that "naturally" arises in bankruptcy problems, we can define the concepts of *Lorenz-Gains* and *Lorenz-Losses* domination.

Given two bankruptcy rules  $f$  and  $g$ , we say that  $f$  *Lorenz-gains dominates*  $g$  if for each  $(E, c) \in \mathcal{B}$ ,  $f(E, c) \succ_L g(E, c)$ . And a bankruptcy rule  $f$  is **Lorenz-Gains Maximal**, **LGM**, if there is no other  $g$  such that, for each  $(E, c) \in \mathcal{B}$ ,

$g(E, c) \succ_L f(E, c)$ . Analogously,  $f$  Lorenz-losses dominates  $g$  if for each  $(E, c) \in \mathcal{B}$ ,  $(c - f(E, c)) \succ_L (c - g(E, c))$ . And a bankruptcy rule  $f$  is **Lorenz-Losses Maximal, LLM**, if there is no other  $g$  such that, for each  $(E, c) \in \mathcal{B}$ ,  $(c - g(E, c)) \succ_L (c - f(E, c))$ .

To clarify these concepts, Figure 6 shows the *Lorenz-Gains* and *Lorenz-Losses Maximal* rules for a bankruptcy problem (Schummer and Thomson [47]). Specifically, the blue line represents the most evenly rule when sharing the estate which, in this case, corresponds with the *Constrained Equal Awards* rule. Analogously, the green line represents the *Constrained Equal Losses* rule, which recommends the most evenly distribution of the losses.

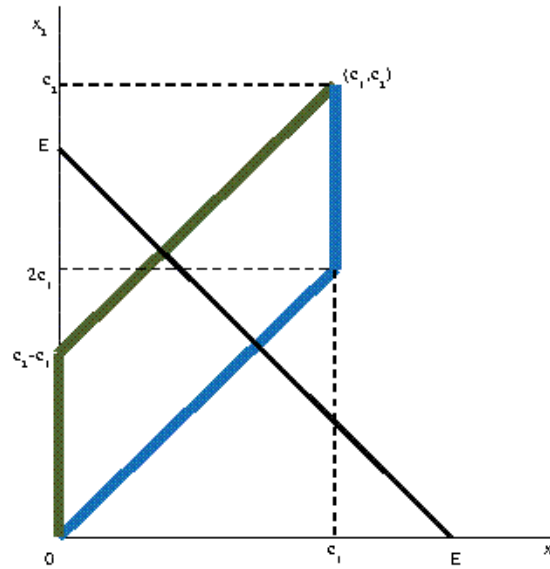


Figure 6: *Lorenz* domination in bi-personal bankruptcy problems.

#### 0.4 Strategic Approach.

As stated, there are three methods to study bankruptcy problems. Whereas the Axiomatic Approach treats these problems from a rules-based viewpoint, the

Strategic Approach (initiated by van Damme [58]) considers rationing problems from a positive viewpoint. That is, the rule which is applied to distribute the resource is not imposed by an external agent, but it is the result of a game where agents act individually pursuing their best strategy, which is that one ensuring her highest payoff. Thus, this analysis is based on the definition of a game where its fundamental requirement is that the recommendation has to be self-enforcing, i.e., there is no external mechanism which let the commitments.

Particularly, in the current work we focus on static games with complete information, denoted by  $\Gamma$ , which are represented as follows,

- a) A set of agents,  $N = \{1, \dots, i, \dots, n\}$ .
- b) A set of strategies  $S_i$  for each agent,  $i \in N$ .
- c) A payoff function  $\pi_i : S_1 \times \dots \times S_n$ , for each agent,  $i \in N$ , which is based on some conjecture concerning the remaining players' strategies,  $s_j \in S_j, j \neq i \in N, .$

Hence, we denote by  $\Gamma$  the strategic game where each agent,  $i \in N$ , chooses a strategy  $s_i \in S_i$ , according to her payoff function  $\pi_i$ ,

$$\Gamma = \{N, \{S_i\}_{i=1}^n, \{\pi_i\}_{i=1}^n\}.$$

Specifically, in such a strategic game we look for a *Nash equilibrium*, which in Nash's [40] words:

*"By using the principles that a rational prediction should be unique, that the players should be able to make use of it, and that such knowledge on the part of each player of what to expect the others to do should not lead him to act out of conformity with the prediction, one is led to the concept."*

This concept, which is the main solution concept in the Strategic Approach, says us that this equilibrium is formed by the optimal strategies for each player, given the other players' strategies. In other words, a *Nash equilibrium* is a situation where there are no player who wants to deviate unilaterally. Formally:

**Theorem** (*Nash, 1951*) *Given a game  $\Gamma$  in strategic form, the profiles  $s^* \equiv (s_1^*, s_2^*, \dots, s_n^*)$  is called a **Nash equilibrium** if for each  $i \in N$  and each  $s_i \in S_i$ ,  $\pi_i(s^*) \geq \pi_i(s_i, s_{-i}^*)$ .*

### 0.5 Transferable Utility Games.

Now we present the third method for analyzing bankruptcy problems: the Cooperative Game Approach, which was initiated by, on the one hand, Dagan and Volij [14], who associate a bargaining problem to each bankruptcy problem and analyze the relationship among solutions; and, on the other hand, O'Neill's [42], who interprets the relationship between bankruptcy problems and a particular class of Transferable Utility Cooperative Games. Concretely, in the current dissertation we will focus in the latter viewpoint. In this sense, O'Neill provides new basis for the resolution of rationing problems using the idea of coalition, which implies that cooperation among agents is allowed. That is, now the rule which is applied to distribute the resource is consequence of a game where groups of players tend to enforce cooperative behavior. Next, we present the main concepts used in this approach.

A particular kind of cooperative games, on what we will pay attention from now on, is the family of transferable utility games, or TU games. In these games, the utility functions for all the agents are chosen so that the rate of transfer of utility among any two of them is 1:1. Therefore, given a set of agents  $N$ , a *TU* game

involving  $N$  can be described as a function  $V$  associating a real number to each subset of agents, or coalition,  $S$  contained in  $N$  (von Neumann and Morgenstern [60]). Formally:

*A TU game is a pair  $(N, V)$ , where  $V : 2^N \rightarrow \mathbb{R}_+$ , and  $V(\emptyset) = 0$ .*

Hereinafter, and because along the present dissertation we do not consider changes in the agents' population, we summarize a TU-game by  $V$ .

Given a coalition  $S \subseteq N$ ,  $V(S)$  is commonly called its worth, and denotes the awards that agents in  $S$  can guarantee by working together. An imputation of a TU game is a sharing of the worth of the grand coalition. Formally:

Let  $(N, V)$  be a TU game. The **set of imputations**, denoted by  $I(N, V)$ , is

$$I(N, V) = \left\{ x \in \mathbb{R}^n : \sum_{i \in N} x_i = V(N) \right\}.$$

Assuming superadditivity, that is, the worth of a coalition is bigger than the sum of the worth of any partition of such a coalition, we can describe a solution for TU games as a correspondence selecting, for each TU game a set of shares of the worth of the grand coalition. Formally:

A **TU game solution** is a correspondence, from a family of TU games,  $G$ , referred to a fixed set of agents,  $N$ ,  $\psi : G \rightarrow \mathbb{R}_+^n$  such that if  $x \in \psi(N, V)$

$$\sum_{i \in N} x_i = V(N).$$

When a TU game solution is univalued, i.e., it consists of a unique share of  $V(N)$ , then we call it a TU game value.

A **TU-value** is a function, from a family of TU-games,  $G$ , referred to a fixed set of agents,  $N$ ,  $\gamma : G \rightarrow \mathbb{R}_+^n$ , such that for each TU-game  $(N, V) \in G$ ,

$$\sum_{i \in N} \gamma_i(V) = V(N).$$

An important kind of TU-games are the *convex games*, since they have a "good" behavior, but before presenting them, we need some additional notation.

Hereinafter, given a TU-game  $(N, V)$ , for each agent  $i \in N$  and each coalition  $S \subset N \setminus i$ , we call the **marginal contribution of agent  $i$  to coalition  $S$** , denoted by  $\Delta_i V(S)$ , the amount which its adherence contributes to the value of the coalition, which is

$$\Delta_i V(S) = V(S \cup \{i\}) - V(S).$$

A TU-game is convex if agents obtain more returns through cooperation, which means that the larger the coalition that agents join, the larger their marginal contribution. Formally: a TU-game  $(N, V)$  is **convex** if and only if, for all  $i \in N$ ,

$$\Delta_i V(S) \leq \Delta_i V(T) \text{ for all } S \subseteq T \subseteq N \setminus \{i\}.$$

Next, we present, among all the TU game solutions in the economic literature, the ones that occupy a prominent place: the *Core*, the *Shapley value* and the *Nucleolus*.

The *Core* (Gillies [19]) is the set of all feasible outcomes (payoffs) that no player (agent) or group of players (coalition) can improve upon by acting for themselves.

Thus, once an agreement in the core has been reached, no individual and no group could gain by regrouping.

Let  $(N, V)$  be a TU game, the **Core** of such a game, denoted by  $\mathbb{C}(N, V)$ , is

$$\mathbb{C}(N, V) = \left\{ x \in \mathbb{R}_+^n : \sum_{i=1}^n x_i = V(N), \sum_{i \in S} x_i \geq V(S) \forall S \subset N \right\}.$$

The *Shapley value* (Shapley [48]) is based on the agent's expected payoff of her marginal contribution to a coalition. Suppose the agents (the elements of  $N$ ) agree to meet at a specified place and time. Naturally, because of a random fluctuations, all arrive at different times; it is assumed, however, that all orders of arrival (permutations of the players) have the same probability  $(1/n!)$ . Suppose that if a player,  $i$ , arrives and finds the members of the coalition  $S - \{i\}$  already there, he receives the amount  $V(S) - V(S - \{i\})$ , i.e., the marginal amount which he contributes to the coalition, as payoff. Then, the *Shapley value* is the expected payoff to player  $i$  under this randomization scheme. Formally:

Let  $(N, V)$  be a TU game, the **Shapley value**,  $Sh_i(N, V)$ , is for an agent  $i$  the expected payoff of her marginal contribution to a coalition:

$$Sh_i(N, V) = \sum_{\substack{S \subset N \\ i \in S}} \frac{(s-1)!(n-s)!}{n!} [V(S) - V(S - \{i\})].$$

where  $s$  and  $n$  are the cardinalities of  $S$  and  $N$ , respectively.

The *Nucleolus* (Schmeidler [45]) minimizes the grade of insatisfaction that a coalition gives to their members, given a particular set of payoff vectors, named excess vector. Formally:

For a TU-game  $(N, V)$  and payoff  $x = (x_1, \dots, x_n)$ , we define the  $2^n$ -vector  $\theta(x)$ , called the **excess vector**, as the vector whose components are the excesses of  $2^n$  subsets  $S \subset N$ , arranged in decreasing order, i.e.,

$$\theta_k(x) = e(S_k, x),$$

where  $S_1, \dots, S_{2^n}$  are the subsets of  $N$ , arranged by

$$e(S_k, x) \geq e(S_{k+1}, x),$$

where for all  $S \subset N$  and  $x = (x_1, \dots, x_n)$ ,  $e(S, x) = V(S) - \sum_{i \in S} x_i$ .

Let  $(N, V)$  be a TU game, the **Nucleolus**,  $Nu(N, V)$ , is

$$Nu(N, V) = \{x \in I(N, V) : \theta(x) <_{Le} \theta(y), \forall y \in I(N, V)\},$$

where  $Le$  denotes the Lexicographic order:

$$\theta(x) <_{Le} \theta(y) \iff \exists k : \{\theta_m(x) = \theta_m(y), \forall m < k, \theta_k(x) < \theta_k(y)\}.$$

At this point, let us introduce the bankruptcy game proposed by O'Neill. His device was to associate to each coalition the quantity of good, if any, that remained left after paying the debts that the bankrupted contracted with all her creditors outside this coalition. Formally:

Given  $(E, c) \in \mathcal{B}$ , the **O'Neill cooperative game** induced by  $(E, c)$  is the pair  $(N, V_{(E,c)}^o)$  where the function  $V_{(E,c)}^o : 2^N \rightarrow \mathbb{R}_+$  associates to each coalition  $S \subseteq N$ , the real number

$$V_{(E,c)}^o(S) = \max \left\{ 0, E - \sum_{i \in N \setminus S} c_i \right\}.$$

This game has been analyzed by Curiel et al. [13], who, firstly, note some "links" between the Axiomatic and the Cooperative Game Approaches to bankruptcy

problems. In this sense, they provide the condition for the correspondence between bankruptcy rules and TU values of the O'Neill's game, known as *Invariance Under Claims Truncation*. It requires that a rule recommends the same allocation if the part of a claim that is above the estate is ignored, or not. Formally:

**Theorem** (Curiel et al., 1987) *A necessary and sufficient condition for a bankruptcy rule  $\varphi$  to be a TU value is*

$$\varphi(E, c) = \varphi(E, \tilde{c}),$$

where the  $i$ -th component of  $\tilde{c}$ , the vector of truncated claims, is

$$\tilde{c}_i(E, c) = \min \{c_i, E\}$$

They also identify the *Core* of such games,

$$C(N, V_{(E,c)}^O) = \left\{ x \in \mathbb{R}^n : \sum_{i=1}^n x_i = V(N), 0 \leq x_i \leq c_i \right\},$$

and show that these bankruptcy games are always *convex*.

Regarding to this, the following results show the correspondence among rules and TU game solutions.

**Theorem** (O'Neill, 1982) *The Random Arrival rule<sup>1</sup> corresponds with the Shapley value.*

**Theorem** (Aumann and Maschler, 1985) *The Talmud rule corresponds with the Nucleolus.*

**Theorem** (Dutta and Ray, 1989) *The Constrained Equal Awards rule corresponds with the Egalitarian Allocation<sup>2</sup>.*

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<sup>1</sup> See O'Neill [42] for a formal definition of this value.

<sup>2</sup> See Dutta and Ray [17] for a formal definition of this value.

**Theorem** (*Curiel et al., 1987*) *The Adjusted Proportional rule<sup>3</sup> corresponds with the  $t$ -value<sup>4</sup>.*

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<sup>3</sup>See *Curiel et al. [13]* for a formal definition of this rule.

<sup>4</sup>See *Tijs [55, 56]* for a formal definition of this value.

## CHAPTER 1

## THE MODEL: LORENZ-BIFOCAL BANKRUPTCY PROBLEMS.

As Moulin [38] points out,

*"Justice is blind, and fairness requires anonymous rules of arbitration. Equals should be treated equally, and unequals, unequally, in proportion to the relevant similarities and differences" (Nicomachean Ethics). This Aristotle's maxim is one of the basis for the formal definition of distributive fairness".*

In this regard, in bankruptcy problems there is no clear answer to the question about what would be a fair way of dividing the resources among the claimants. Economic Theory attempts to clarify the fairness criterion behind rules by seeking the agreement on appealing properties or requirements which should be fulfilled by the acceptable allocation (see for instance, O'Neill [42], Dutta and Ray [17], Young [63, 64], and Gura and Maschler, [20]).

Henceforth, we can draw that when a society has to solve a bankruptcy problem, it may be quite natural that it establishes a *"Commonly Accepted Equity Principles"* set on the allowed rules, trying to find the commitment and the fairest distribution. Moreover, note that this idea of establishing restrictions on the possible allocations has been prospected deeply in the context of Nash's bargaining model (see for instance, van Damme's [58], Chun [8], Marco et al. [31], Naeve-Steinweg [39], and Thomson [53]).

However, as Young [65] highlights:

*"Fairness does not boil down to a single formula, but represents a balance between competing principle of need, desert and social utility".*

Note that he is telling us that when a rule has to be proposed to distribute the resource, society should not only care about the individual awards but also about the social benefits, i.e., each agent concerns for individual and social welfare. In this regard, as we have mentioned in Section 0.3 of Preliminaries, many authors have considered that the general social goal is to treat everybody as evenly as possible (see for instance Arin [4] and Dutta and Ray [17]). At this point, and following Cowell [12] and Lambert [26], among all the equality criteria that appear in the literature on egalitarianism, we decide to consider as general goal equality in terms of the *Lorenz-criterion*, i.e., an allocation is more equally distributed than another, if the former *Lorenz dominates* the latter.

In this work, we pick up all these ideas to introduce extended bankruptcy problems, named *Lorenz-Bifocal Bankruptcy Problems*.

Firstly, we consider a "*Commonly Accepted Equity Principles*" set,  $P_l$ , which must be fulfilled by all the proposed rules for bankruptcy problems.

Secondly, the discrepancy for sharing the resources is considered by means of the existence of two fixed rules, that is, two focus which gather the two most egalitarian rules according to extreme and opposite ways of facing bankruptcy problems: awards and losses (Section 0.1 of Preliminaries).

We call these two focus **Lorenz-Focal rules**, which are the *Lorenz-Gains* and *Lorenz-Losses Maximal rules* satisfying  $P_l$ , denoted by  $LGM^{P_l}$  and  $LLM^{P_l}$ , respectively.

Finally, a *Lorenz-Bifocal Bankruptcy Problem* is the triplet consisting of a bankruptcy problem and the two **Lorenz-Focal rules** according to a fixed "*Legitimate Principles*" set,  $P_l$ . Formally:

**Definition 3** *A Lorenz-Bifocal Bankruptcy Problem,  $LB_{P_l}$ , is a triplet  $LB_{P_l} = (B, LGM^{P_l}, LLM^{P_l})$  where  $B = (E, c) \in \mathcal{B}$ ,  $P_l$  is a fixed set of principles on which a particular society has agreed, and both  $LGM^{P_l}$  and  $LLM^{P_l}$  are the Lorenz-Gains and Lorenz-Losses Maximal rules satisfying  $P_l$  for each  $B \in \mathcal{B}$ , respectively.*

Let  $P$  be the set of all subsets of properties of rules. Each  $P_l \in P$  represents a specific society which will always apply such principles for solving its problems, and  $\mathcal{LB}_P$  denotes the set of all *Lorenz-Bifocal Bankruptcy Problems*.

Now, the question is

*How can we distribute the resources for each Lorenz-Bifocal Bankruptcy Problem?*

If we think in a "natural" manner, the answer arise as follows. If a society faces a bankruptcy problem with two focal rules which are interpreted as acceptable extremes and opposite viewpoints to solve the problem, it seems logical to consider as *Admissible* rules those ones satisfying the set of required principles that assign, to each agent, an allocation between the focus.

So that, a *Lorenz-Bifocal Admissible rule* will be a rule satisfying  $P_l$  which recommends an allocation between the two *Lorenz-Focal rules*. Formally:

**Definition 4** A *Lorenz-Bifocal Admissible rule* on  $\mathcal{LB}_P$  is a function  $\varphi : \mathcal{LB}_P \rightarrow \mathbb{R}^n$ , such that for each Lorenz-Bifocal bankruptcy problem  $LB_{P_l} \in \mathcal{LB}_P$ ,  $\varphi$  is a rule satisfying  $P_l$ , associating for each  $i \in N$  a part of the resources satisfying

$$\begin{aligned} \min\{LGM_i^{P_l}(E, c), LLM_i^{P_l}(E, c)\} &\leq \varphi_i(E, c) \\ &\leq \max\{LGM_i^{P_l}(E, c), LLM_i^{P_l}(E, c)\}. \end{aligned}$$

Let  $\Phi$  denote the set of all rules and let  $\Phi(LB_{P_l})$  be the subset of *Lorenz-Bifocal Admissible rules* in  $P_l$ .

In this context, we consider three possible choices of "*Commonly Accepted Equity Principles*" set that a society could impose on the rules, taking into account that the introduced properties have been understood by many authors as minimal requirements of fairness (see for instance Thomson [52]). Specifically, we consider the following "*Legitimate Principles*" sets,

$$P_1 = \{\text{Efficiency, Claim-boundedness and Non-Negativity}\},$$

$$P_2 = P_1 \cup \{\text{Resource Monotonicity, Super-Modularity and Midpoint Property}\},$$

and

$$P_3 = P_1 \cup \{\text{Resource Monotonicity and Midpoint Property}\}.$$

Given these sets of equity principles, Schummer and Thomson [47] showed that the *Constrained Equal Awards* rule is the only *Lorenz-Gains Maximal* for  $P_1$ . Moreover, by Bosmans and Lauwers [6] we know that *Piniles'* rule is the only *Lorenz-Gains Maximal* rule for  $P_2$ . Finally, Chun, Schummer and Thomson [8] obtained that the *Constrained Egalitarian* rule is the only *Lorenz-Gains Maximal* rule for  $P_3$ . Next, all these results are presenting:

**Theorem** (*Schummer and Thomson, 1997*) For each  $(E, c) \in \mathcal{B}$ , the *Constrained Equal Awards* rule is the only one in  $\Phi(P_1)$  such that: (i) the gap between the smallest and the largest amount any claimant receives is the smallest, and (ii) the variance of the amounts received by all claimants is the smallest.

**Theorem** (*Bosmans and Lauwers, 2007*) For each  $(E, c) \in \mathcal{B}$ , *Piniles'* rule is the only one in  $\Phi(P_2)$  such that: (i) the gap between the smallest and the largest amount any claimant receives is the smallest, and (ii) the variance of the amounts received by all claimants is the smallest.

**Theorem** (*Chun, Schummer and Thomson, 2001*) For each  $(E, c) \in \mathcal{B}$ , *Piniles'* rule is the only one in  $\Phi(P_3)$  such that: (i) the gap between the smallest and the largest amount any claimant receives is the smallest, and (ii) the variance of the amounts received by all claimants is the smallest.

Starting from these results and using the concept of dual bankruptcy rule, it can be deduced directly that *Constrained Equal Losses*, the *Dual of the Piniles'* and the *Dual Constrained Egalitarian* rules are the only *Lorenz-Losses Maximal* for  $P_1$ ,  $P_2$ , and  $P_3$ , respectively.

Hence, the *Lorenz-Focal rules* that mark out the region of *Lorenz-Bifocal Admissible rules* for  $P_1$ ,  $P_2$ , and  $P_3$ , are the pairs  $(CEA, CEL)$ ,  $(Pin, DPin)$ , and  $(CE, DCE)$ , respectively. So, *Lorenz-Bifocal Bankruptcy Problems* for each of these principles sets are well-defined, being their elements triplets, such that, for each  $(E, c) \in B$ ,

$$LB_{P_1} = ((E, c), CEA, CEL),$$

$$LB_{P_2} = ((E, c), Pin, DPin), \text{ and}$$

$$LB_{P_3} = ((E, c), CE, DCE).$$

In this context, the following figures show the area of the associated *Lorenz-Bifocal Admissible rules* for  $P_1$ ,  $P_2$  and  $P_3$ .

In Figure 1.1 the blue and the green lines represent the two *Lorenz-Focal rules* for  $P_1$ , corresponding with the *CEA* and *CEL* rules, respectively. Any *Lorenz-Bifocal Admissible rule* in this set, must satisfy  $P_1$  and be placed in the green area.

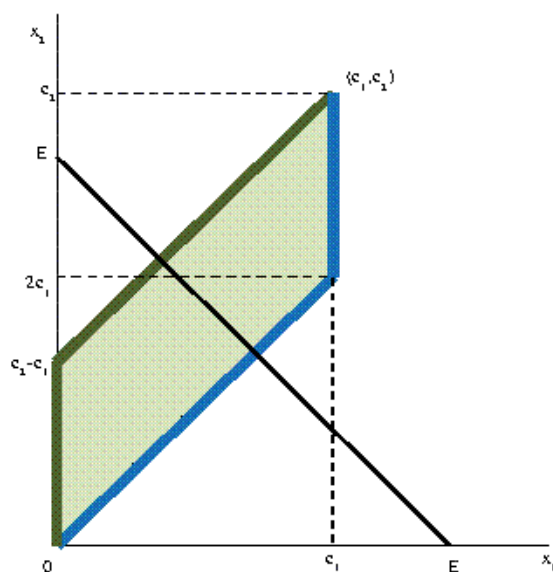


Figure 1.1: *Lorenz-Bifocal Admissible rules* for  $P_1$ .

Figure 1.2 shows the two *Lorenz-Focal rules* for  $P_2$ , represented by the blue and the green lines, which correspond with *Pin* and *DPin* rules, respectively. Any *Lorenz-Bifocal Admissible rule* in this set, must satisfy  $P_2$  and be placed in the green area.

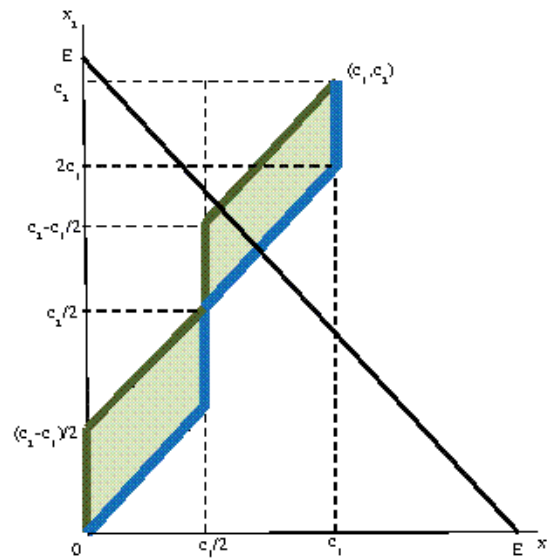


Figure 1.2: *Lorenz-Bifocal Admissible rules* for  $P_2$ .

Finally, in Figures 1.3 and 1.4 the blue and the green lines represent the two *Lorenz-Focal rules* for  $P_3$ , corresponding with the *CE* and *DCE* rules, respectively. The first figure when the relation between agent 1 and 2 when the distance between their claims is lower than the smallest agent's claim. The second figure otherwise. Any *Lorenz-Bifocal Admissible rule* in this set, must satisfy  $P_3$  and be placed in the green area.

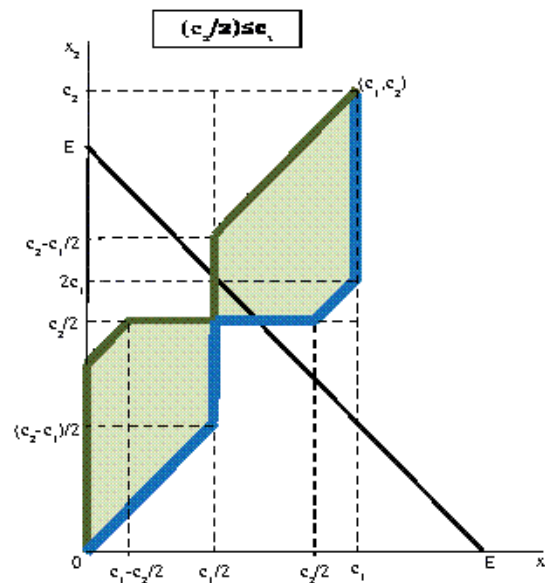


Figure 1.3: *Lorenz-Bifocal Admissible rules for  $P_3$  (case a).*

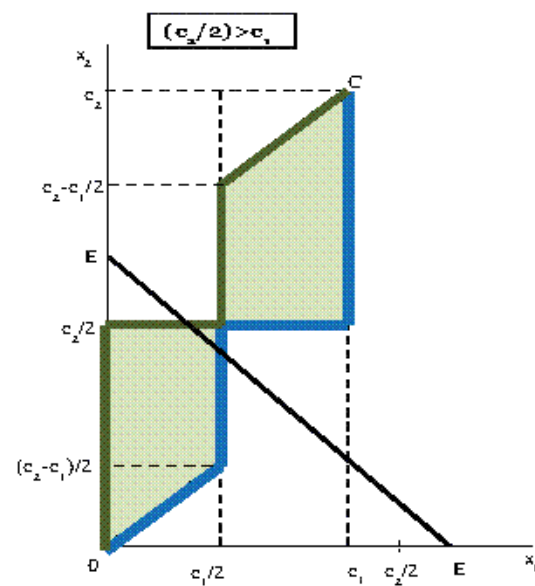


Figure 1.4: *Lorenz-Bifocal Admissible rules for  $P_3$  (case b).*

## CHAPTER 2

## A NEW APPROACH FOR BOUNDING AWARDS.

## 2.1 Introduction.

As we have already mentioned (see Section 0.2 of Preliminaries), many authors have introduced certain lower bounds on awards in bankruptcy problems: O'Neill [42], Herrero and Villar [23, 24], Moulin [37], Moreno-Ternero and Villar [33] and Dominguez [15]. However, most of these guarantees have only been justified by their own reasonability. A clear exception is the condition *Respect of Minimal Rights*, which requires that each claimant receives at least what is left of the endowment after the other claimants have been fully compensated, or zero if this amount is negative. This property is not only intrinsically appealing, but also, as Thomson [52] pointed out, it is a consequence of efficiency, non-negativity and claim-boundedness together (See Definition 2).

Our goal is to propose a new method by which the idea of lower bound on awards arises not "ad hoc" but, in a natural way, in the context of *Lorenz-Bifocal Bankruptcy Problems*. In other words, we recover the O'Neill's line, establishing restrictions on awards which determine the *Lorenz-Bifocal Admissible rules* for each *Lorenz-Bifocal Bankruptcy Problem*. Then, we define the agent's *Lorenz P-Safety* as the smallest amount she receives among all the allowed rules. Finally, we define the associated lower bound on awards, *Respect of Lorenz P-Safety*, by demanding that each agent should receive at least her *Lorenz P-Safety*.

Since, in general, the aggregate guaranteed amount by means of the *Lorenz P-Safety* will not exhaust the endowment, we propose and analyze its recursive application, called the *Recursive Lorenz P-Safety Process*. The idea of recursivity is not new, and indeed has already been used for introducing bankruptcy rules by Alcalde et al. [1], who generalize Ibn Ezra's proposal, and by Dominguez and Thomson [16], who propose the *Recursive* rule by using Moreno-Tertero and Villar's concept of boundedness.

We apply the previous methodology to the three different sets of properties or "*Legitimate Principles*" previously introduced. Firstly, we analyze the set  $P_1$ , all rules are allowed, finding that the *Recursive Lorenz P-Safety* rule leads to the *Constrained Equal Losses* rule. We then consider the set  $P_2$ , consisting of *Resource Monotonicity*, *Super-Modularity* and *Midpoint Property*. In this society, we show that the *Recursive Lorenz P-Safety* rule leads to the *Dual of Piniles'* rule.

Previous results could be written as follows:

*"In the set of all the acceptable rules according to both  $P_1$  and  $P_2$ , the recursive application of the Lorenz P-Safety recovers one of its extremes: the one providing greater awards to the higher claimants".*

Then, analyzing the generalization of this statement for any  $P_t$ , naturally arises as a question. With this aim, we consider the set of socially accepted requirements,  $P_3$ . Surprisingly, since  $P_2 \subset P_3 \subset P_1$ , we show both that this generalization is not possible and that the rule obtained by the recursive application of the *Lorenz P-Safety* does not satisfy all of the equity principles on which this process is based.

The chapter is organized as follows: Section 2.2 proposes our new approach for bounding awards and its recursive application. Section 2.3 provides a new basis to

classical bankruptcy rules using the previous ideas. Section 2.4 shows a non existence result of the proposed process for the set  $P_3$ . Section 2.5 presents the dual results. Section 2.6 summarizes the conclusions. Finally, technical proofs are contained in Appendix A.

## 2.2 The model.

In this section, taking *Lorenz-Bifocal Bankruptcy Problems* as a starting point, we propose a new lower bound on awards based on the application of the ordinary meaning of a guarantee, named the *Lorenz P-Safety*. For each *Lorenz-Bifocal Bankruptcy Problem*, it provides each agent the smallest amount recommended for her by all the *Lorenz-Bifocal Admissible rules* satisfying the selected properties. Taking into account that each *Lorenz-Bifocal Admissible rule* has to recommend an allocation between the two *Lorenz-Focal rules*, our lower bound ensures each agent the minimum amount between them. Formally:

**Definition 5** Given  $LB_{\rho_i} \in \mathcal{LB}_{\rho}$ , the **Lorenz P-Safety**,  $Ls_i$ , is for each  $i \in N$ ,

$$Ls_i(LB_{\rho_i}) = \min \{ LGM_i^{\rho_i}(E, c), LLM_i^{\rho_i}(E, c) \}.$$

In general, since the sum of the agents' *Lorenz P-Safeties* of a problem does not exhaust the endowment, a requirement of composition from the profile of these bounds arises in a natural way: the awards vector should be equivalently obtainable either directly or by first assigning to each agent her *Lorenz P-Safety*, adjusting claims down by these amounts, and finally, applying the rule to divide the remainder. The following definition applies this idea.

**Definition 6** Given  $LB_{\rho_i} \in \mathcal{LB}_{\rho}$ , a rule  $\varphi$  satisfies **Lorenz P-Safety First** if for each  $(E, c) \in \mathcal{B}$  and each  $i \in N$ ,

$$\varphi_i(E, c) = Ls_i(LB_{\rho_i}) + \varphi_i\left(E - \sum_{i \in N} Ls_i(LB_{\rho_i}), c - Ls(LB_{\rho_i})\right).$$

This kind of composition, or equivalently applying a recursive method from a lower bound on awards, has been used to generate new rules. The *Recursive* rule, proposed by Dominguez and Thomson [16], is the result of applying this procedure with the *Securement* lower bound. Following this idea, we define the recursive application of our *Lorenz P-Safety*, which will be called the *Recursive Lorenz P-Safety Process*.

**Definition 7** For each  $m \in \mathbb{N}$ , the *Recursive Lorenz P-Safety Process* at the  $m$ -th step, *RLS*, associates for each  $LB_{\rho_i} \in \mathcal{LB}_{\rho}$ , and each  $i \in N$ ,

$$[RLS(LB_{\rho_i}^m)]_i = Ls_i(LB_{\rho_i}^m),$$

$$\text{where } LB_{\rho_i}^m \equiv ((E^m, c^m), LGM^{\rho_i}, LLM^{\rho_i}),$$

$$(E^1, c^1) \equiv (E, c) \text{ and for } m \geq 2,$$

$$(E^m, c^m) \equiv (E^{m-1} - \sum_{i \in N} Ls_i(LB_{\rho_i}^{m-1}), c^{m-1} - Ls(LB_{\rho_i}^{m-1})).$$

According to this process, at the first step an agent will receive her *Lorenz P-Safety* of the original problem. We then define a residual problem in which the endowment is what remains and the claims are adjusted down by the amounts just given. All agents then receives her *Lorenz P-Safety* of this residual problem, and so on. In general, it can not be ensured that the sum of the amounts that agents receive in each step exhausts the endowment, but when this occurs, we call this sum the *Recursive Lorenz P-Safety* rule. In this regard, note that non-negativity and claim boundedness are satisfied by construction. Furthermore, by adapting the proof of Remark 3 in Appendix A, it is clear that whenever the *Lorenz P-Safety* provides a positive amount to some agent in each step, efficiency is met. In this regard, note that the recursive application of the *Minimal Rights* fails this requirement, since from the second step on, no agent receives anything. The result of combining this procedure with the aforementioned bound, therefore, would not be a rule.

**Definition 8** The *Recursive Lorenz P-Safety* rule,  $\varphi^{RL}$ , associates for each  $LB_{P_i} \in \mathcal{LB}_P$ , and each  $i \in N$ ,  $\varphi_i^{RL}(LB_{P_i}) = \sum_{m=1}^{\infty} [RLS^m(LB_{P_i})]_i$ , whenever

$$\sum_{i \in N} \left( \sum_{m=1}^{\infty} [RLS^m(LB_{P_i})]_i \right) = E.$$

Note that this rule will be a *Lorenz-Bifocal Admissible* rule whenever it fulfills the set of equity principles on which the *Recursive Lorenz P-Safety Process* is based. A fact that, as we will see later, cannot always be guaranteed.

At this point we should mention some contributions that have certain features in common with our previous approach, although in the context of Nash's bargaining model. In this sense, a bargaining game is a pair  $(Q, d)$ , where the feasible set,  $Q \subset \mathbb{R}^n$ , consists of all utility vectors attainable by the group  $N$  by unanimous agreement, and the disagreement point,  $d$ , which is a point of  $Q$ , is the utility vector that results if they fail to reach an agreement. In this regard, a bargaining solution is a function defined on a class of bargaining games that associates with each game in the class a unique point in the feasible set of the game. In this context, Marco et al. [31] define the *Unanimous-Concession* mechanism, as follows. At a first step, their process guarantees to each agent the smallest amount according to a set of agreed solutions, which would be the disagreement point at the following step, and so on. Thomson [53] discusses an approach where, starting from a class of solutions on which agents have agreed, all the feasible payoff vectors that are not chosen by some members are eliminated and the solutions are reapplied until convergence is reached. He introduces and studies the concept of *Closedness Under Recursion* of a family of solutions, which means that the solution obtained is not only well-defined but also belongs to the family of solutions considered. This idea, although in a different framework, is close to our definition of *Admissible* rule, but the process he uses has no relation to ours.

### 2.3 The Recursive Application of the Lorenz P-Safety for $P_1$ and $P_2$ .

In this section, we provide the *Recursive Lorenz P-Safety* rule when the "Commonly Accepted Equity Principles" sets are  $P_1$  and  $P_2$ .

As we have seen, for  $P_1 = \{\text{Efficiency, Claim-boundedness and Non-Negativity}\}$ , we are not requiring any additional property different from those imposed by the definition of a rule. Thus, the *Lorenz-Focal rules* will be the *Lorenz-Gains* and *Lorenz-Losses Maximal* elements in the set of all bankruptcy rules, which, as we know by Schummer and Thomson [47], are given by the pair  $(CEA, CEL)$ . So that, the associated *Lorenz-Bifocal Bankruptcy Problem* is,  $LB_{P_1} = ((E, c), CEA, CEL)$ , where  $(E, c) \in \mathcal{B}$ ; and the *Lorenz-Bifocal Admissible rules* are those rules such that for each  $i \in N$ , and each  $(E, c) \in \mathcal{B}$ ,  $\min\{CEA_i(E, c), CEL_i(E, c)\} \leq \varphi_i^{L_{P_1}}(E, c) \leq \max\{CEA_i(E, c), CEL_i(E, c)\}$ .

Then, for each *Lorenz-Bifocal Bankruptcy Problem*  $LB_{P_1} = ((E, c), CEA, CEL)$ , its associated *Lorenz P-Safety*,  $Ls$ , comes straightforwardly from our previous definition, that is, for each agent  $i \in N$ ,  $Ls_i(LB_{P_1}) = \min\{CEA_i(E, c), CEL_i(E, c)\}$ .

This *Lorenz P-Safety* is represented, for the bi-personal case, in Figure 2.1. The black line represents the estate  $E$ . The blue and the green lines show  $CEA$  and  $CEL$  rules, which are the two *Lorenz-Focal rules* marking out the area of all the *Lorenz-Bifocal Admissible rules* for  $P_1$ . Moreover, given  $E$ , the point  $Ls$  shows the *Lorenz P-Safety* for agents 1 and 2.

As we can see, the smallest amount that agent 1 can receive is the one recommended by the  $CEL$  rule. For agent 2, her *Lorenz P-Safety* is the amount recommended by  $CEA$  rule.

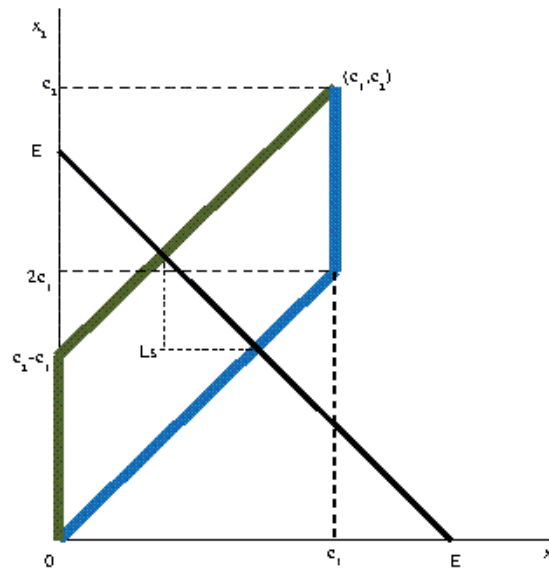


Figure 2.1: *Lorenz P-Safety* for  $P_1$ .

Next theorem tells us that the *Recursive Lorenz P-Safety* rule for  $P_1$  recovers one of the two extreme rules; specifically, the one which favors the highest claimant.

**Theorem 1** For each  $LB_{P_1} \in \mathcal{LB}_{P_1}$ ,  $LB_{P_1} = ((E, c), CEA, CEL)$ , the *Recursive Lorenz P-Safety* rule is the *Constrained Equal Losses* rule,  $\varphi^{RL}(LB_{P_1}) = CEL(E, c)$ .

**Proof.** See Sections A.1 and A.2, in Appendix A. ■

Note that previous result can be generalized as follows. Let  $P_1^*$  such that,  $P_1 \subset P_1^*$  and such that the properties for  $P_1^* \setminus P_1$  are satisfied by both the *Constrained Equal Awards* and the *Constrained Equal Losses* rules. Then, for each  $LB_{P_1^*} \in \mathcal{LB}_{P_1^*}$ , the *Lorenz P-Safety* will remain the smallest amount recommended by the same

*Lorenz-Focal rules*. And, consequently, the *Recursive Lorenz P-Safety* rule will be the *Constrained Equal Losses* rule,  $\varphi^{RL}(LB_{P_1}) = CEL(E, c)$ .

In this regard, note that, as we comment in Chapter 6, if we add to this set of equity principles the properties *Order Preservation* and *Resource Monotonicity*, we will obtain the same results, so that, they are redundant in our context.

Next, we analyze the application of our process to the set  $F_2$  which, in addition to *claim-boundedness*, *non-negativity* and *efficiency*, contains *Resource Monotonicity*, *Super-Modularity* and *Midpoint Property*.

In this set, we know, by Bosmans and Lauwers [6] and duality, that the two *Lorenz-Focal rules* that mark out the region of *Lorenz-Bifocal Admissible rules* for  $F_2$  are given by the pair  $(Pin, DPin)$ . So that, the associated *Lorenz-Bifocal Bankruptcy Problem* is,  $LB_{P_2} = ((E, c), Pin, DPin)$ , where  $(E, c) \in \mathcal{B}$ ; and the *Lorenz-Bifocal Admissible rules* are those rules satisfying the accorded properties and such for each  $i \in N$ , and each  $(E, c) \in \mathcal{B}$ ,  $\min\{Pin_i(E, c), DPin_i(E, c)\} \leq \varphi_i^{LB_{P_2}}(E, c) \leq \max\{Pin_i(E, c), DPin_i(E, c)\}$ .

Henceforth, applying Definition 5, for each *Lorenz-Bifocal Bankruptcy Problem*  $LB_{P_2} = ((E, c), Pin, DPin)$ , its associated *Lorenz P-Safety*,  $Ls$ , for each agent,  $i \in N$ , is  $Ls_i(LB_{P_2}) = \min\{Pin_i(E, c), DPin_i(E, c)\}$ .

In Figure 2.2, which shows the previous situation for bi-personal problems, the black line the estate  $E$ . The blue and the green lines show  $Pin$  and  $DPin$ , respectively,

which are the two *Lorenz-Focal rules* marking out the area of all the *Lorenz-Bifocal Admissible rules* for  $P_2$ .

Moreover, given  $E$ , the point  $Ls$  shows the *Lorenz P-Safety* for agents 1 and 2, which coincide with the amount recommended by the *DPin* and *Pin* rules, respectively.

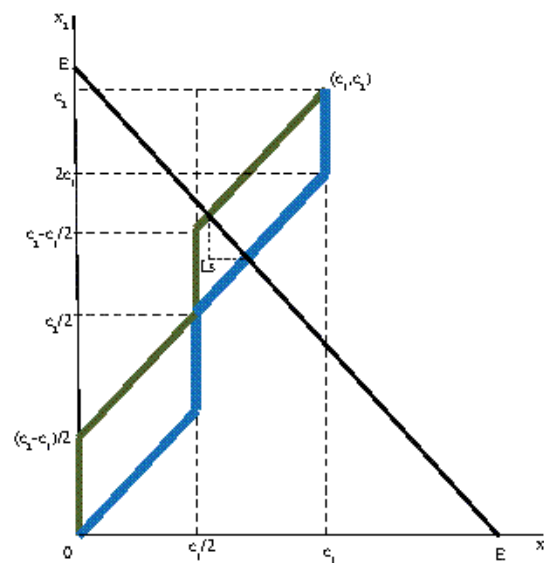


Figure 2.2: *Lorenz P-Safety* for  $P_2$ .

As previously, next result shows that the *Recursive Lorenz P-Safety* rule recovers, again, the *Lorenz-Focal rule* that favors the highest claimant. Analogously, this result can be generalized considering any set,  $P_2^*$ , of "Equity principles" which contains  $P_2$  ( $P_2 \subset P_2^*$ ) and such that the properties for  $P_2^* \setminus P_2$  are satisfied by both the *Piniles'* and the *Dual of Piniles'* rules. Then, for each  $LB_{P_2^*} \in \mathcal{LB}_P$ , the *Lorenz-Focal rules*

will remain the same. Hence, its associated *Lorenz P-Safety* will not change and the *Recursive Lorenz P-Safety* rule will be the *Dual of Piniles'* rule.

**Theorem 2** *For each  $LB_{P_2} \in \mathcal{LB}_P$ ,  $LB_{P_2} = ((E, c), Pin, DPin)$ , the Recursive Lorenz P-Safety rule is the Dual of Piniles' rule,  $\varphi^{RL}(LB_{P_2}) = DPin(E, c)$ .*

**Proof.** See Sections A.1 and A.3, in Appendix A. ■

To conclude this section, let us remark that, for the two sets of properties we are considering, the associated *Lorenz P-Safety* is defined as the minimum of both a classical rule and its dual. These rules are extreme and opposite ways of sharing awards among claimants in the corresponding set of allowed rules. The first intuition when applying the recursive procedure is to get some allocation in the middle of these rules. However, we have proved that the *Recursive Lorenz P-Safety* rule leads to one of the *Lorenz-Focal rules*, namely that one favoring the largest claims. Our results can therefore be interpreted as providing a new basis for old rules. Moreover, they lead to a natural question, which is analyzed in the next section:

"For any appealing set of equity principles, would its *Lorenz P-Safety recursive application* recover one of the two *Lorenz-Focal rules*?"

## 2.4 A negative result.

In this section we show that, in general, the recursive application of our new lower bound on awards does not provide one of the *Lorenz-Focal rules* satisfying the considered set of equity principles. Surprisingly, the rule obtained by our procedure does not always satisfy the properties on which it is based; that is, it is not a *Lorenz-Bifocal Admissible rule*. Let us consider the set of equity principles  $P_3$ , which is an

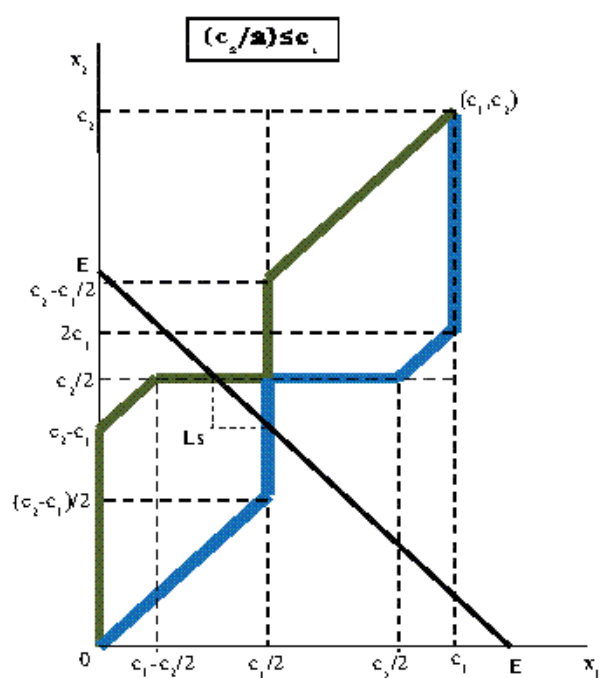
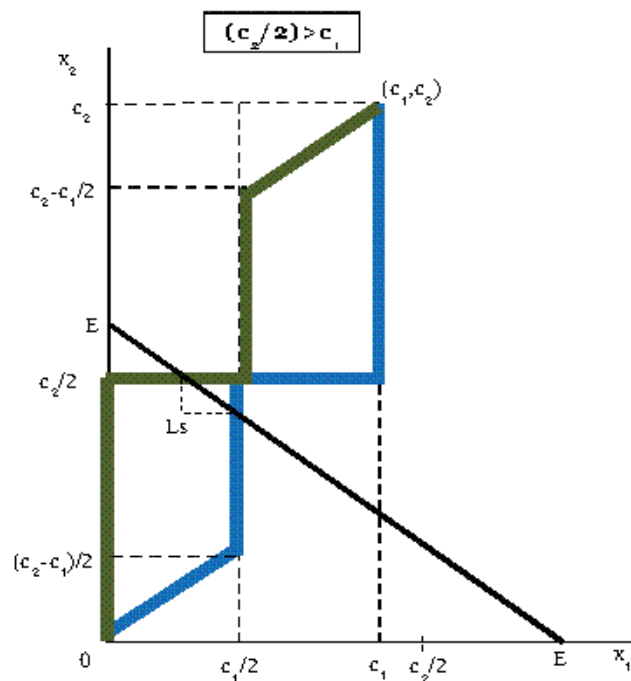
"intermediate" situation between  $P_1$  and  $P_2$ , since is more restrictive than  $P_1$ , we require *Resource Monotonicity* and *Midpoint Property*, but more permissive than  $P_2$ , since we eliminate *Super-Modularity*.

Applying results in Chun, Schummer and Thomson [8] and duality, the two extreme rules that mark out the region of allowed rules for  $P_3$  are given by the pair  $(CE, DCE)$ . So that, the associated *Lorenz-Bifocal Bankruptcy Problem* is,  $LB_{P_3} = ((E, c), CE, DCE)$ , where  $(E, c) \in \mathcal{B}$ . The *Lorenz-Focal rules* are those satisfying the accorded properties such that for each  $i \in N$ , and each  $(E, c) \in \mathcal{B}$ ,  $\min\{CE_i(E, c), DCE_i(E, c)\} \leq \varphi_i^{L_{P_3}}(E, c) \leq \max\{CE_i(E, c), DCE_i(E, c)\}$ .

Then, for each *Lorenz-Bifocal Bankruptcy Problem* with  $P_i = P_3$ ,  $LB_{P_3} \in \mathcal{LB}_P$ ,  $LB_{P_3} = ((E, c), CE, DCE)$ , the *Lorenz P-Safety*,  $Ls$ , comes straightforwardly from Definition 5, that is, for each agent,  $i \in N$ ,  $Ls_i(LB_{P_3}) = \min\{CE_i(E, c), DCE_i(E, c)\}$ .

Figures 2.3 and 2.4 show this situation for bi-personal problems. In these figures, the black solid line represents all the possible sharing of the estate  $E$ . The blue and green lines show  $CE$  and  $DCE$ , which are the two *Lorenz-Focal rules* marking out the area of all *Lorenz-Bifocal Admissible rules* for  $P_3$ . The point  $Ls$  shows the *Lorenz P-Safety* for agents 1 and 2, which coincides with the amount recommended by the  $DCE$  and  $CE$  rules, respectively.

In this context, we show that, although for bi-personal problems the *Recursive Lorenz P-Safety* rule for  $P_3$  retrieves the *Dual Constrained Egalitarian* rule, this fact can not be extended for the n-person case.

Figure 2.3: Lorenz P-Safety for  $P_3$  (case a).Figure 2.4: Lorenz P-Safety for  $P_3$  (case b).

**Theorem 3** For each bi-personal  $LB_{P_3} \in \mathcal{LB}_P$ ,  $LB_{P_3} = ((E, c), CE, DCE)$ , the Recursive Lorenz  $P$ -Safety rule is the Dual Constrained Egalitarian rule,  $\varphi^{RL}(LB_{P_3}) = DCE(E, c)$ .

**Proof.** See Sections A.1 and A.4, in Appendix A. ■

**Proposition 1** There is a problem,  $(E, c) \in \mathcal{B}$ , for which the Recursive Lorenz  $P$ -Safety rule for  $P_3$  does not coincide with the Dual Constrained Egalitarian rule,  $\varphi^{RL}(LB_{P_3}) \neq DCE(E, c)$ .

**Proof.** See Sections A.1 and A.5, in Appendix A. ■

Our next proposition highlights the fact that the composition of "appealing" equity principles and "natural" processes for finding solutions does not always guarantee desirable results. In particular, we find that the society should be careful when establishing its equity principles, if the recursive process is applied, since the Recursive Lorenz  $P$ -Safety rule so obtained cannot be *Lorenz-Bifocal Admissible*.

**Proposition 2** For  $P_3$ , the Recursive Lorenz  $P$ -Safety rule does not satisfy Resource Monotonicity.

**Proof.** See Sections A.1 and A.5, in Appendix A. ■

Finally, next corollary comes from a complete overview of previous results, and it points out the following fact: Our procedure's success can not even be ensured for a set of properties that is, at the same time, an enlargement and a reduction of other sets for which this process is a *Lorenz-Bifocal Admissible* rule.

**Proposition 3** Let  $P_j, P_k$  and  $P_l$  in  $P$ , such that  $P_j \subset P_k \subset P_l$  and such that for each  $(E, c) \in \mathcal{B}$ ,  $LB_{P_j} = (B, LGM^{P_j}, LLM^{P_j})$  and  $LB_{P_l} = (B, LGM^{P_l}, LLM^{P_l})$  in  $\mathcal{LB}_P$ ,  $\varphi^{RL}(LB_{P_j})$  and  $\varphi^{RL}(LB_{P_l})$  are *Lorenz-Bifocal Admissible* rules. It cannot then be ensured that  $\varphi^{RL}(LB_{P_k})$  is a *Lorenz-Bifocal Admissible* rule.

### 2.5 Duality.

The previous analysis can be rewritten for losses by using the idea of duality. When focusing on losses, the starting point will be the *Dual Lorenz-Bifocal Bankruptcy Problems*. That is, given  $P_t \in P$  and  $(E, c) \in \mathcal{B}$ , we consider the set of dual properties  $(P_t)^d$  and the problem of distributing the losses  $(L, c)$ .

Given a *Lorenz-Bifocal Bankruptcy Problem*,  $LB_{P_t}$ , the *Dual Lorenz-Bifocal Bankruptcy Problem*, denoted by  $(LB_{P_t})^d$ , is a triplet  $(LB_{P_t})^d = (B^d, LGM^{(P_t)^d}, LLM^{(P_t)^d})$  where  $B^d = (L, c) \in \mathcal{B}$ ,  $P_t$  is a fixed set of principles on which a particular society has agreed, and both  $LGM^{(P_t)^d}$  and  $LLM^{(P_t)^d}$  are the *Lorenz-Gains* and *Lorenz-Losses Maximal* rules satisfying  $(P_t)^d$  for each  $B^d \in \mathcal{B}$ , respectively.

Note that, since all the considered properties of  $P_t$ , with  $t = \{1, 2, 3\}$ , are *Self-Dual* we remain using the same sets of equity principles. So that, in this context, it can be shown that the *Constrained Equal Awards* and *Piniles'* rules are retrieved for  $P_1$  and  $P_2$ , respectively. However, we cannot obtain a generalized result for  $P_3$ .

**Corollary 1** For each  $LB_{P_1} \in \mathcal{LB}_P$ ,  $(LB_{P_1})^d = ((L, c), CEA, CEL)$ , the *Recursive Lorenz P-Safety* rule is the *Constrained Awards Losses* rule,  $\varphi^{RL} \left( (LB_{P_1})^d \right) = CEA(E, c)$ .

**Corollary 2** For each  $LB_{P_2} \in \mathcal{LB}_P$ ,  $(LB_{P_2})^d = ((L, c), Pin, DPin)$ , the *Recursive Lorenz P-Safety* rule is the *Piniles'* rule,  $\varphi^{RL} \left( (LB_{P_2})^d \right) = Pin(E, c)$ .

Note that previous result can be generalized as follows.

Let  $P_1^*$  such that,  $P_1 \subset P_1^*$  and such that the properties for  $P_1^* \setminus P_1$  are satisfied by both the *Constrained Equal Awards* and the *Constrained Equal Losses* rules. Then, for each  $LB_{P_1^*} \in \mathcal{LB}_P$ , the *Lorenz P-Safety* will remain the

smallest amount recommended by the same *Lorenz-Focal rules*. Consequently, the *Recursive Lorenz P-Safety* rule on losses will be the *Constrained Equal Awards* rule,  $\varphi^{RL}((LB_{P_1^*})^d) = CEA(E, c)$ .

Let  $P_2^*$  such that,  $P_2 \subset P_2^*$  and such that the properties for  $P_2^* \setminus P_2$  are satisfied by both *Piniles'* and the *Dual of Piniles'* rules. Then, for each  $LB_{P_1^*} \in \mathcal{LB}_P$ , the *Lorenz P-Safety* will remain the smallest amount recommended by the same *Lorenz-Focal rules*. Consequently, the *Recursive Lorenz P-Safety* rule on losses will be *Piniles'* rule,  $\varphi^{RL}((LB_{P_1^*})^d) = Pin(E, c)$ .

**Corollary 3** For each bi-personal  $LB_{P_3} \in \mathcal{LB}_P$ ,  $(LB_{P_3})^d = ((L, c), CE, DCE)$ , the *Recursive Lorenz P-Safety* rule is the *Constrained Egalitarian* rule,  $\varphi^{RL}((LB_{P_3})^d) = CE(E, c)$ .

**Corollary 4** There is a problem,  $(E, c) \in \mathcal{B}$ , for which the *Recursive Lorenz P-Safety* rule for  $P_3$  does not coincide with the *Constrained Egalitarian* rule,  $\varphi^{RL}((LB_{P_3})^d) \neq CE(E, c)$ .

Finally, note that, in this context, we can treat this *Lorenz P-Safety* on losses as an upper bound. Particularly, in the following chapter, we interpret this bound as the highest amount that each agent can receive according with all the *Lorenz-Bifocal Admissible rules* for  $P_l$ .

## 2.6 Conclusions.

We have continued a research line that underlies theory of bankruptcy problems from their very beginning: the search for a "fair" minimum amount that each agent should receive in each situation. In this context, our main contribution is a new method for bounding awards based on the *Lorenz-Focal rules*.

Our main results have been obtained by applying the methodology to different sets of equity principles which have been interpreted as "basic" requirements

by many authors. For two possible societies, permissive and restrictive, we have retrieved, respectively, the *Constrained Equal Losses* and the *Dual of Piniles'* rules when focusing on awards; and the *Constrained Equal Awards* and *Piniles'* rules when focusing on losses. Nevertheless, by considering a society whose "*Commonly Accepted Equity Principles*" set is included in the permissive society's set and includes the restrictive society's set, we have shown that the *Recursive Lorenz P-Safety* rule does not satisfy one of the equity principles upon which such a society initially agreed to found its decisions. Thus, the need for in-depth study of the consequences of social agreements on both principles and procedure has been emphasized, as they could become meaningless when combined.

To sum up, this chapter: (i) offers the understanding of old bankruptcy rules from a new viewpoint; and (ii) warns of the difficulties that may be involved in composing "a priori" appropriate pieces of a puzzle.

## CHAPTER 3

## CONVEX COMBINATION OF BANKRUPTCY RULES: A JUSTIFICATION.

## 3.1 Introduction.

In this chapter we propose the simultaneous combination of both a lower and an upper bound on awards when facing *Lorenz-Bifocal Bankruptcy Problems*. To do this, note that, as we have pointed out, rationing problems can be interpreted from a double viewpoint: awards and losses. With the former case we focus on the amount of gains that we get. In this regard, we can find everyday situations where each agent has guaranteed some amount of the endowment, such as, nowadays, most of the governments have assured an amount of money deposited on the people's bank accounts. On the other hand, these problems can be faced bothering about the quantity of incurred losses or, in other words, we concern on the highest level of awards that we can obtained. For example, and returning to the previous illustration, some governments have established an upper bound on the bank managers' salaries.

It does not therefore seem unreasonable to think that each agent should receive at least the smallest amount according to all the allowed rules which satisfy some accorded principles, on the one hand, and on the other, no more than the highest amount provided by all these rules.

Now, by using as starting point the *Lorenz-Bifocal Bankruptcy Problems*, we define the *Lorenz Double Boundedness Recursive Process* as the recursive double imposition of a lower and an upper bound on awards, and show that the rule so

obtained coincides with the average of the *Lorenz-Focal rules*. This result has two consequences. Firstly, we provide a new justification of the convex combination of two extreme and opposite ways of distributing the endowment. Secondly, we obtain a new method which is invariant to the viewpoint (gains and losses) used to ration the resources. This means that our new approach treats symmetrically the problem of "what is available" and "what is missing", i.e., the rule so obtained is *Self-Dual*.

Finally, we specify these results to the three different sets of equity principles,  $P_i \in \{P_1, P_2, P_3\}$ , providing new basis for the average of old bankruptcy rules.

The chapter is organized as follows: Section 3.2 proposes our new approach. The main results and their application on the three sets of properties are provided in Section 3.3. Section 3.4 summarizes our conclusions. Finally, Appendix B gathers technical proofs.

### 3.2 The model.

In this section we define an upper bound on awards, named the *Lorenz P-Ceiling*, as the highest amount that each agent could get according to the application of the *Lorenz-Bifocal Admissible rules* satisfying the "*Commonly Accepted Equity Principles*", i.e., our upper bound ensures that each agent's awards are confined to the maximum amount between the *Lorenz-Focal rules* for each  $P_i$ . Formally:

**Definition 9** Given  $LB_{P_i} \in \mathcal{LB}_{P_i}$ , the *Lorenz P-Ceiling*,  $Lc$ , is for each  $i \in N$ ,

$$Lc_i(LB_{P_i}) = \max \{ LGM_i^{P_i}(E, c), LLM_i^{P_i}(E, c) \}.$$

Note that, whenever the accorded properties are *Self-Dual*, if we focussed on losses instead of awards, an alternative way of understanding this limit would be as a lower

bound on losses (see Section 2.5 of previous chapter), i.e., for each  $(LB_{P_i}) \in \mathcal{LB}_P$ , the *Lorenz P-Ceiling*, for each  $i \in N$ , is the smallest loss incurred for each agent by all the *Lorenz-Bifocal Admissible rules* in  $P_i$ ,

$$\begin{aligned} Lc_i(LB_{P_i}) &= c_i - \min \{LGM_i^{P_i}(L, c), LLM_i^{P_i}(L, c)\} \\ &= c_i - Ls_i \left( (LB_{P_i})^d \right). \end{aligned}$$

Therefore, the *Lorenz P-Safety* and the *Lorenz P-Ceiling* are dual, a fact that will be used later on.

Figures 3.1, 3.2, 3.3 and 3.4 represent the two bounds for bi-personal problems with  $P_i \in \{P_1, P_2, P_3\}$ , respectively.

Particularly, the black line represents the estate ( $E$ ). The blue and the green lines show the *Lorenz-Focal* rules marking out the area of all the admissible paths of awards satisfying the properties in  $P_1$ ,  $P_2$ , and  $P_3$ . That is, the green and the blue solid lines are the *CEL* and the *CEA* rules for  $P_1$ , the *DPin* and *Pin* rules for  $P_2$ , and the *DCE* and the *CE* rules for  $P_3$ , respectively.

As we can see, for  $P_1$ , the agent 1's *Lorenz P-Safety* is given by the *CEL* rule, and the *Lorenz P-Ceiling* is given by the *CEA* rule. For agent 2 is the opposite, i.e., the *CEA* rule is her *Lorenz P-Safety* and the *CEL* rule is her the *Lorenz P-Ceiling*.

For  $P_2$ , the agent 1's *Lorenz P-Safety* is given by the *DPin* rule, and the *Lorenz P-Ceiling* is given by *Pin* rule. For agent 2 the *DPin* rule is her *Lorenz P-Ceiling* and *Pin* rule is her *Lorenz P-Safety*.

For  $P_3$ , the agent 1's *Lorenz P-Safety* is given by the *DCE* rule, and the *Lorenz P-Ceiling* is given by the *CE* rule. For agent 2 the *DCE* rule is her *Lorenz P-Ceiling* and the *CE* rule is her *Lorenz P-Safety*.

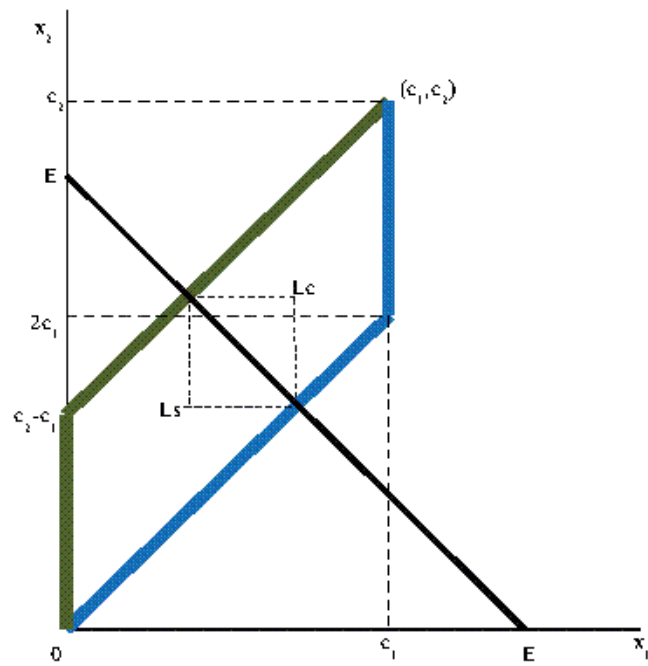


Figure 3.1: *Lorenz P-Safety* and *Lorenz P-Ceiling* for  $P_1$ .

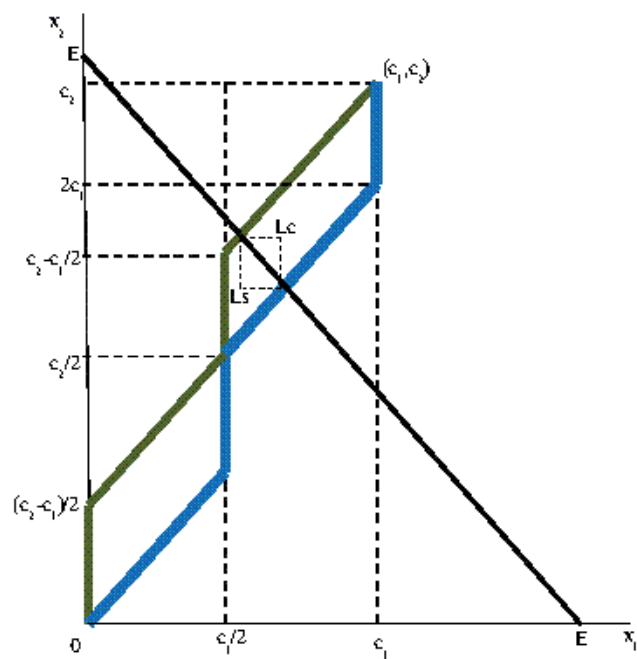


Figure 3.2: *Lorenz P-Safety* and *Lorenz P-Ceiling* for  $P_2$ .

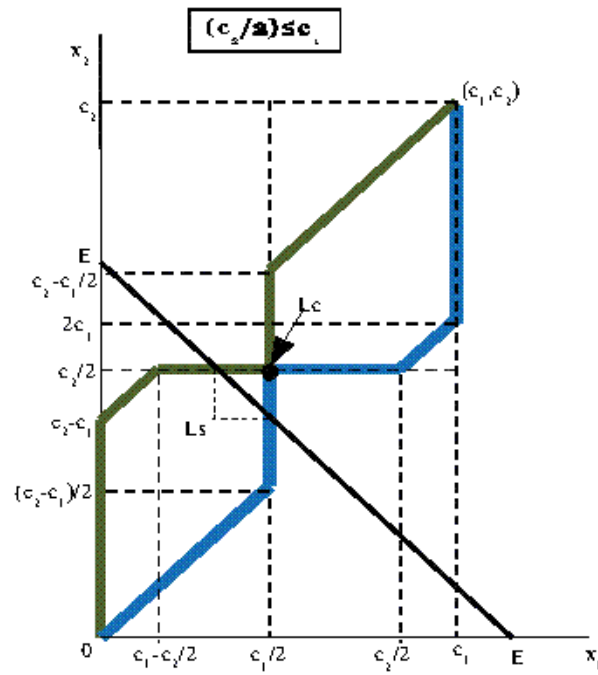


Figure 3.3: *Lorenz P-Safety* and *Lorenz P-Ceiling* for  $P_3$  (case a).

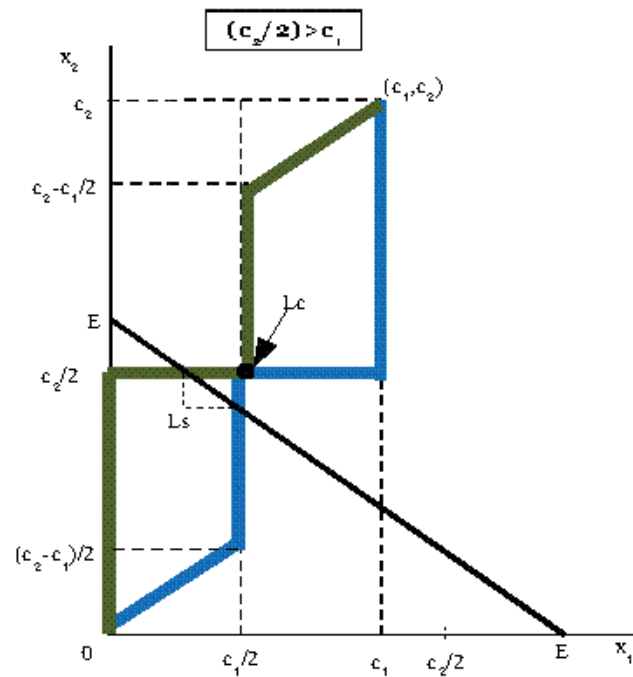


Figure 3.4: *Lorenz P-Safety* and *Lorenz P-Ceiling* for  $P_3$  (case b).

As we have mentioned in the introduction of the current chapter, there are situations in which a society reaches an agreement on the properties which define the way of rationing the endowment, and then decides that each agent should receive, at least, a minimum amount, and, at most, a maximum amount according to all the *Lorenz-Bifocal Admissible rules*. With these ideas in mind, we define the *Lorenz Double Boundedness Recursive Process* as the procedure in which at each step, every agent's claim is truncated by her *Lorenz P-Ceiling*, and each of them receives her *Lorenz P-Safety*.

**Definition 10** Given  $m \in \mathbb{N}$ , the *Lorenz Double Boundedness Recursive Process*, *LDBR*, associates for each  $LB_{\rho_i} \in \mathcal{LB}_\rho$  and each  $i \in N$ ,

$$[LDBR(LB_{\rho_i}^m)]_i = Ls_i(LB_{\rho_i}^m),$$

where  $LB_{\rho_i}^m \equiv ((E^m, c^m), LGM^{\rho_i}, LLM^{\rho_i})$ ,

$$(E^1, c^1) \equiv (E, c) \text{ and for } m \geq 2,$$

$$E^m \equiv E^{m-1} - \sum_{i \in N} Ls_i(LB_{\rho_i}^{m-1}).$$

$$c_i^m = Lc_i(LB_{\rho_i}^{m-1}) - Ls_i(LB_{\rho_i}^{m-1}).$$

According to this process, an agent will get at the first step her *Lorenz P-Safety* of the original problem. At the second step, we redefine a residual problem, in which the endowment consists of the remaining resources and each agent's claim is truncated by her *Lorenz P-Ceiling*, and then adjusted down by the amount received. Then each agent receives her *Lorenz P-Safety* of this residual problem, and so on. We can easily see that this process is not always efficient, but when it does, we call it the *Lorenz Double Recursive rule*. Formally:

**Definition 11** The *Lorenz Double Recursive* rule,  $\varphi^{LDR}$ , associates for each  $LB_{P_i} \in \mathcal{LB}_P$  and each  $i \in N$ ,  $\varphi_i^{LDR}(LB_{P_i}) = \sum_{m=1}^{\infty} [LDBR(LB_{P_i}^m)]_i$ , whenever

$$\sum_{i \in N} \left( \sum_{m=1}^{\infty} [LDBR(LB_{P_i}^m)]_i \right) = E.$$

Note that this rule will be a *Lorenz-Bifocal Admissible* rule whenever it fulfills the set of equity principles on which the *Lorenz Double Boundedness Recursive Process* is based. A fact that, as we will see later, cannot be always guaranteed.

### 3.3 Main results and applications.

This section presents a general result and its specification on the sets of equity principles  $P_1$ ,  $P_2$  and  $P_3$ . Firstly, next theorem establishes that, whenever the properties selected are *Self-Dual*, the final allocation provided by the *Lorenz Double Recursive* rule will correspond with the average of the lower and upper bounds for each *Lorenz-Bifocal Bankruptcy Problem*, i.e., the *Lorenz P-Safety* and the *Lorenz P-Ceiling*.

**Theorem 4** For each  $LB_{P_i} \in \mathcal{LB}_P$ , such that  $P_i$  is *Self-Dual*, then for each  $i \in N$ ,

$$\varphi_i^{LDR}(LB_{P_i}) = \frac{Lc_i(LB_{P_i}) + Ls_i(LB_{P_i})}{2}.$$

**Proof.** See Appendix B.1. ■

Therefore, the next result, which follows straightforwardly from Theorem 4, shows that the *Lorenz Double Recursive* rule for  $P_i$  can be defined as the average of the associated *Lorenz-Focal* rules.

**Proposition 4** For each  $LB_{P_i} \in \mathcal{LB}_P$ , such that  $P_i$  is *Self-Dual*, then

$$\varphi^{LDR}(LB_{P_i}) = \frac{LGM^{P_i}(E, c) + LLM^{P_i}(E, c)}{2}.$$

A direct consequence of the above proposition is that the *Lorenz Double Recursive rule* not only is always well defined, i. e. satisfies efficiency, non-negativity and claim-boundedness, but it also satisfies *Self-Duality* (Thomson and Yeh [54]), which means that it provides the same allocation of the endowment when distributing awards or losses.

Moreover, with this result we retrieve the convex combination of the two extreme *Lorenz-Focal rules*. Particularly, the *Lorenz Double Recursive rule* proposes the midpoint between the two rules which represent extreme and opposite ways of sharing awards among claimants according to the imposed requirements. So, in other words, it could be said that the rationing of the endowment obtained by the recursive double imposition of the *Lorenz P-Safety* and the *Lorenz P-Ceiling* neither favor nor hurts to any agent in particular. Following Thomson and Yeh [54]:

*"When two rules express opposite viewpoints on how to solve a bankruptcy problem, it is natural to compromise between them by averaging".*

Concluding this section, we specify the previous result for the three different sets of equity principles mentioned previously.

From Chapter 2 we know that with  $P_l \in \{P_1, P_2, P_3\}$ , the *Lorenz-Focal rules* which delimit the region of *Lorenz-Bifocal Admissible rules* are well defined. We have also obtained the *Lorenz P-Safety* for each of these sets of equity principles, as well as the *Lorenz P-Ceiling*, by duality. It is therefore a straightforward process of applying Theorem 4 in order to obtain the following results.

**Corollary 5** *For each  $LB_{P_l} \in \mathcal{LB}_P$ ,  $LB_{P_l} = ((E, c), CEA, CEL)$ , the *Lorenz Double Recursive rule* is the average of the *Constrained Equal Awards* and the *Constrained Equal Losses rules*.*

**Corollary 6** For each  $LB_{P_2} \in \mathcal{LB}_P$ ,  $LB_{P_2} = ((E, c), Pin, DPin)$ , the *Lorenz Double Recursive rule* is the average of *Piniles'* and the *Dual of Piniles'* rules.

**Corollary 7** For each  $LB_{P_3} \in \mathcal{LB}_P$ ,  $LB_{P_3} = ((E, c), CE, DCE)$ , the *Lorenz Double Recursive rule* is the average of the *Constrained Egalitarian* and the *Dual Constrained Egalitarian* rules.

Note that these corollaries imply that the allocation proposed by our new procedure is *Lorenz-Bifocal Admissible* for  $P_i \in \{P_1, P_2, P_3\}$ , since, the *Lorenz Double Recursive rule* preserves *Resource Monotonicity*, *Super-Modularity* and the *Midpoint* property.

However, the *Lorenz Double Recursive rule* fails some properties such that *Composition Down* (Moulin [36]), *Composition Up* (Young [65]), and *Consistency* (Young [65]), even when the two *Lorenz-Focal rules* fulfill them. It is thus not possible to ensure that the final allocation is *Lorenz-Bifocal Admissible* when the *Lorenz P-Safety* and the *Lorenz P-Ceiling* are used recursively at the same time. For example, it can be observed that if the property of *Consistency* is added to the set  $P_1$ , the two *Lorenz-Focal rules* are still the *Constrained Equal Awards* and the *Constrained Equal Losses* rules, but the average of these *Lorenz Admissible* rules does not satisfy one of the initial properties. The natural question to pose, therefore, is "*Would a society have any argument to apply this new rule?*" It could, because although a society may also have reasons for not agreeing to apply the result of the *Lorenz Double Recursive rule*, as it may fail some properties, this way of distributing the endowment has been defended by many authors as a natural way to agree on a "middle" allocation between two extreme ways of rationing (see for instance Thomson and Yeh [54]). Another reason to use this method is that we know exactly the result of the procedure and the properties which are satisfied by the final allocation.

### 3.4 Conclusions.

This chapter defines a new method for distributing the endowment, using the idea of recursively guaranteeing 'fair' minimum and maximum amounts to each agent in bankruptcy problems, known as the *Lorenz Double Recursive rule*. In this context, our main result states that the *Lorenz Double Recursive rule* corresponds with the average of the two extreme *Lorenz-Focal rules*, implying that our new rule is always well defined and retrieves an allocation which coincides with the midpoint between the rule that favours the highest claimant and the rule that favours the lowest agent, i.e., new justifications are obtained for the convex combination of rules.

Finally, this procedure relates specifically to certain proposed equity principles sets that many authors have interpreted as 'basic' requirements, recovering the average of old and well-known rules. We have also shown that our process does not guarantee that the final allocation will be admissible.

## CHAPTER 4

## STRATEGIC JUSTIFICATIONS FOR OLD BANKRUPTCY RULES.

## 4.1 Introduction.

In this chapter we analyze *Lorenz-Bifocal Bankruptcy Problems* from a non-cooperative viewpoint, by using the *Diminishing Claims* and the *Unanimous Concessions* mechanisms proposed by Chun [8] and Herrero [21], respectively. The starting point of both procedures is the bargaining model introduced by van Damme [58], who prospectively Nash equilibria of a non-cooperative game. Particularly, he defines a mechanism of successive concessions, where agents' strategies consist of the choice of a rule among a reasonable set of them. From van Damme's work, other mechanisms for bargaining and bankruptcy have been proposed (see for instance Naeve-Steinweg [39]). On the one hand, Chun [8] proposes the *Diminishing Claims* procedure, in which each agent's claim is truncated by the maximal amount among all the reasonable rules. On the other hand, Herrero [21] modifies the *Unanimous Concessions* mechanism, provided by Marco et al. [31] (introduced in Chapter 2) ensuring to each agent the minimal amount that all the proposed rules concede to her.

Following this line, and more recently, García-Jurado et al. [18] propose an elementary game where each agent's strategy belongs to a determined closed space of possible choices. With this game, they show that any acceptable rule can be obtained as the unique allocation of the corresponding Nash equilibria depending on its associated closed interval of strategies. Finally, Herrero et al. [22] provide an

experimental strategic support to the *Proportional* rule, showing that this one is the choice of most of the players.

Next, we apply this methodology in the framework of the *Lorenz-Bifocal Bankruptcy Problems* and analyze its application on the three sets of equity principles,  $P_1$ ,  $P_2$  and  $P_3$ , introduced previously. Firstly, we start with the prospection of the *Unanimous Concessions* mechanism. In this case, Herrero [21] finds out that in any Nash equilibrium induced by such a game, the *Constrained Equal Losses* rule is recommended for  $P_1$ , and we prove that the *Dual of Piniles'* rule is the one recommended for  $P_2$ . Moreover, when using  $P_3$ , we show that the applications of this procedure does not always provide desirable distributions. Secondly, applying duality, the *Constrained Equal Awards* rule is recovered (Chun [8]) in any Nash equilibrium induced by the *Diminishing Claims* procedure for  $P_1$ . Moreover, we obtain that, for  $P_2$ , *Piniles'* rule is retrieved. Finally, we define a new mechanism consisting on the simultaneous application of both procedures, and we provide a non-cooperative justification of the convex combination of two extreme and opposite ways of distributing the endowment.

The chapter is organized as follows: Section 4.2 presents the *Unanimous Concessions* procedure and its associated results. Section 4.3 provides the results obtained when applying the *Diminishing Claims* procedure. Section 4.4 proposes and analyses the simultaneous application of the previous mechanisms. Section 4.5 summarizes our conclusions. Finally, Appendix C gathers technical proofs.

## 4.2 The Unanimous Concessions procedure.

In this section, we start defining the *Unanimous Concessions* procedure (Marco et al. [31]; Herrero [21]) which works as follows. Given that agents have chosen their preferred admissible rules, if at the initial step there is no agreement, at the second

step, each agent receives the minimal amount among all the proposed. Now, we redefine the residual bankruptcy problem, in which the endowment is the leftover resource, the claims are adjusted down by the amounts just given, and the same procedure is applied. The solution will be the limit of the procedure if it is feasible, and zero otherwise. Formally:

**Definition 12** *Unanimous Concessions procedure,  $u$ , (Herrero, 2003):*

Let  $LB_{P_i} \in \mathcal{LB}_P$ . At the first stage, each agent chooses a rule  $\varphi^i \in \Phi(LB_{P_i})$ . Let  $\psi = (\varphi^i)$  be the profile of selected rules. The proposal of the Unanimous Concessions procedure,  $u[\psi, LB_{P_i}]$  is obtained as follows, for each  $m \in \mathbb{N}$ :

[Step 1] If all agents agree on  $\varphi(LB_{P_i})$ , then  $u[\psi, LB_{P_i}] = \varphi(LB_{P_i})$ . Otherwise, go to next step.

[Step 2] Let us define

$$Ls_i(LB_{P_i}) = \min_{j \in N} \varphi_i^j(LB_{P_i}),$$

$$c^2 = c - Ls(LB_{P_i}),$$

$$E^2 = E - \sum_{i \in N} Ls_i(LB_{P_i}), \text{ and}$$

$$LB_{P_i}^2 \equiv ((E^2, c^2), LGM^{P_i}, LLM^{P_i}).$$

Now, if all agents agree on  $\varphi(LB_{P_i}^2)$ , then  $u[\psi, LB_{P_i}] = Ls(LB_{P_i}) + \varphi(LB_{P_i}^2)$ .

Otherwise, go to next step.

[Step  $m + 1$ ] Let us define

$$Ls_i(LB_{P_i}^m) = \min_{j \in N} \varphi_i^j(LB_{P_i}^m),$$

$$c^{m+1} = c^m - Ls(LB_{P_i}^m),$$

$$E^{m+1} = E^m - \sum_{i \in N} Ls_i(LB_{P_i}^m), \text{ and}$$

$$LB_{P_i}^{m+1} \equiv ((E^{m+1}, c^{m+1}), LGM^{P_i}, LLM^{P_i}).$$

Now, if all agents agree on  $\varphi(LB_{P_i}^{m+1})$ , then  $u[\psi, LB_{P_i}] = \sum_{k=1}^m Ls(LB_{P_i}^k) + \varphi(LB_{P_i}^{m+1})$ . Otherwise, go to next step.

[Limit case] Compute  $\sum_{k=1}^{\infty} Ls(LB_{\rho_i}^k)$ . If it converges to an allocation,  $x$ , such that  $\sum_{i \in N} x_i \leq E$ ,  $u[\psi, LB_{\rho_i}] = x$ . Otherwise,  $u[\psi, LB_{\rho_i}] = 0$ .

From now on, let  $\Gamma_{LB_{\rho_i}}^u$  denote the game induced by the *Unanimous Concessions* procedure when agents act strategically, in which the set of players is  $N$ , the strategies for each agent are rules in  $\Phi(LB_{\rho_i})$  and the payoffs are the sum of the amounts received by each agent in each step  $m \in \mathbb{N}$ . That is,

$$\Gamma_{LB_{\rho_i}}^u = \left\{ N, \left\{ \varphi^i \in \Phi(LB_{\rho_i}) \right\}_{i=1}^n, \left\{ \sum_{k=1}^m Ls_i(LB_{\rho_i}^k) \right\}_{i=1}^n \right\},$$

where  $m$  denotes the step where the agreement is reached, and  $\infty$  otherwise.

Next, we consider the "*Commonly Accepted Equity Principles*" sets  $P_1$  and  $P_2$  previously introduced, to apply this procedure:

$P_1$ , which does not require any additional property to those fulfilled by any rule, that is, *claims boundedness*, *non negativity* and *efficiency*.

$P_2$ , which contains *Resource Monotonicity*, *Super-Modularity* and the *Midpoint Property*.

As Herrero [21] points out, in all non-cooperative Nash equilibria induced by *Unanimous Concessions* procedure for  $P_1$ , each agent will receive the awards recommended by the *Constrained Equal Losses* rule.

**Theorem 5** (Herrero, 2003) *In the game  $\Gamma_{LB_{\rho_i}}^u$ , the CEL rule is a weakly dominant strategy for the agent with the highest claim. In any Nash equilibrium induced by the game  $\Gamma_{LB_{\rho_i}}^u$ , each agent receives the amount given by the CEL rule.*

Now, next propositions tell us that, when the set of equity principles is  $P_2$ , if some agent announces the *DPin* rule, then the *Unanimous Concessions* procedure

converge to this rule. Moreover, the *DPin* rule is a weakly dominant strategy for the agent with the highest claimant. Then, as a direct consequence of these two results, we show that in all non-cooperative Nash equilibria, each agent will receive the awards recommended by the *DPin* rule.

**Proposition 5** *For each  $LB_{P_2} \in \mathcal{LB}_P$ ,  $LB_{P_2} = ((E, c), Pin, DPin)$ , and each  $i \in N$ , if  $\varphi^i(E, c) \in \Phi(LB_{P_2})$ , and for some  $j \in N$ ,  $\varphi^j(E, c) = DPin(E, c)$ , then  $u[\psi, LB_{P_2}] = DPin(E, c)$ .*

**Proof.** See Sections C.1 and C.2 in Appendix C. ■

**Proposition 6** *In the game  $\Gamma_{LB_{P_2}}^u$ , the *DPin* rule is a weakly dominant strategy for the agent with the highest claim.*

**Proof.** See Sections C.1 and C.3 in Appendix C. ■

**Theorem 6** *In any Nash equilibrium induced by the game  $\Gamma_{LB_{P_2}}^u$ , each agent receives the amount given by the *DPin* rule.*

**Proof.** See Sections C.1 and C.4 in Appendix C. ■

Finally, it is interesting to observe that in both cases  $P_1$  and  $P_2$ , the sequence defined by the *Unanimous Concessions* procedure converges to an efficient point, and, moreover, it recovers one of the two *Lorenz-Focal rules*, specifically, the one which favors the highest claimants. So, again, as in Chapter 2 we can ask,

"Would the *Unanimous Concessions* procedure recover one of the two *Lorenz-Focal rules* for any appealing set of equity principles?"

## 4.2.1 A NEGATIVE RESULT.

In the next results we show that, in general, the Nash equilibrium induced by the application of the *Unanimous Concessions* procedure does not provide one of the *Lorenz-Focal rules*. Furthermore, the allocation proposed by it fails one of "*Commonly Accepted Equity Principles*" on which the process is based. With this purpose, we use the set of equity principles  $P_3$ , in which we eliminate *Super-Modularity* from  $P_2$ . That is,  $P_3$  asks for *Resource Monotonicity* and the *Midpoint Property*.

In this context, and for the bi-personal games, the following proposition tells us that if some agent announces the *DCE* rule, then the *Unanimous Concessions* procedure converge to this rule.

**Proposition 7** *For each  $LB_{\rho_3} \in \mathcal{LB}_{\rho}$ ,  $LB_{\rho_3} = ((E, c), CE, DCE)$ , with  $|N| = 2$ , and each  $i \in \{1, 2\}$ , if  $\varphi^i(E, c) \in \Phi(LB_{\rho_3})$ , and for some  $j \in \{1, 2\}$ ,  $\varphi^j(E, c) = DCE(E, c)$ , then  $u[\psi, LB_{\rho_3}] = DCE(E, c)$ .*

**Proof.** See Sections C.1 and C.5 in Appendix C. ■

Finally, we establish that for the bi-personal problems in  $P_3$  each agent receives the awards recommended by the *DCE* rule in the Nash equilibrium induced by the *Unanimous Concessions* procedure.

**Theorem 7** *For each  $LB_{\rho_3} \in \mathcal{LB}_{\rho}$ ,  $LB_{\rho_3} = ((E, c), CE, DCE)$ , with  $|N| = 2$ , in any Nash equilibrium induced by the game  $\Gamma_{LB_{\rho_3}}^u$ , each agent receives the amount given by the *DCE* rule.*

**Proof.** See Sections C.1 and C.6 in Appendix C. ■

Nevertheless, the above results cannot be extended for more than two agents as we show in the next theorems.

**Theorem 8** *There is a problem,  $(E, c) \in \mathcal{B}$ , for which in a Nash equilibrium induced by the game  $\Gamma_{\mathcal{LB}_{\rho_3}}^u$ , not all agents receive the amount given by the DCE rule.*

**Proof.** See Sections C.1 and C.7 in Appendix C. ■

**Theorem 9** *The Nash equilibrium induced by the game  $\Gamma_{\mathcal{LB}_{\rho_3}}^u$  does not fulfill Resource Monotonicity.*

**Proof.** See Sections C.1 and C.7 in Appendix C. ■

### 4.3 The Diminishing Claims procedure.

As we have already mentioned, bankruptcy problems can be faced from two viewpoints: gains and losses. By using this dual relation between awards and losses (Aumann and Maschler [5]) and the fact that all the proposed properties are *Self-Dual*, the results of the previous section can be analyzed focusing on the maximum awards that each agent can ensure, that is, the minimal losses incurred.

In this sense, the *Diminishing Claims* procedure (Chun [8]), denoted by  $d$ , says that, given that agents have chosen their preferred rules, if at the initial step there is no agreement, at the second step, we redefine the residual bankruptcy problem, in which the endowment does not change, and each agent's claim is truncated by the maximal amount among all the proposed at step 1. Then, the procedure is again applied until an agreement is reached. If this is not the case, the solution will be the limit of the procedure if it is feasible, and zero otherwise. Formally:

**Definition 13** *Diminishing Claims procedure,  $d$ , (Chun, 1989):*

*Let  $LB_{\rho_i} \in \mathcal{LB}_{\rho}$ . At the first stage, each agent chooses a rule  $\varphi^i \in \Phi(LB_{\rho_i})$ . Let  $\psi = (\varphi^i)$  be the profile of selected rules. The proposal of the Diminishing Claims procedure,  $d[\psi, LB_{\rho_i}]$  is obtained as follows, for each  $m \in \mathbb{N}$ :*

[Step 1] If all agents agree on  $\varphi(LB_{P_i})$ , then  $d[\psi, LB_{P_i}] = \varphi(LB_{P_i})$ . Otherwise, go to next step.

[Step 2] Let us define

$$Lc_i(LB_{P_i}) = \max_{j \in N} \varphi_i^j(LB_{P_i}),$$

$$c^2 = Lc(LB_{P_i}),$$

$$E^2 = E, \text{ and}$$

$$LB_{P_i}^2 = ((E^2, c^2), LGM^{P_i}, LLM^{P_i}).$$

Now, if all agents agree on  $\varphi(LB_{P_i}^2)$ , then  $d[\psi, LB_{P_i}] = \varphi(LB_{P_i}^2)$ . Otherwise, go to next step.

[Step  $m+1$ ] Let us define

$$Lc_i(LB_{P_i}^m) = \max_{j \in N} \varphi_i^j(LB_{P_i}^m),$$

$$c^{m+1} = Lc(LB_{P_i}^m),$$

$$E^{m+1} = E, \text{ and}$$

$$LB_{P_i}^{m+1} = ((E^{m+1}, c^{m+1}), LGM^{P_i}, LLM^{P_i}).$$

Now, if all agents agree on  $\varphi(LB_{P_i}^{m+1})$ , then  $d[\psi, LB_{P_i}] = \varphi(LB_{P_i}^{m+1})$ . Otherwise, go to next step.

[Limit case] Compute  $\lim_{k \rightarrow \infty} \varphi(LB_{P_i}^k)$ . If it converges to an allocation,  $x$ , such that  $\sum_{i \in N} x_i \leq E$ ,  $d[\psi, LB_{P_i}] = x$ . Otherwise,  $d[\psi, LB_{P_i}] = 0$ .

Hereinafter, let  $\Gamma_{LB_{P_i}}^d$  denote the game induced by the *Diminishing Claims* procedure when agents acts strategically, in which the set of players is  $N$ , the strategies for each agent are rules in  $\Phi(LB_{P_i})$  and the payoffs are the amount recommending to each agent by the accorded rule. That is,

$$\Gamma_{LB_{P_i}}^d = \left\{ N, \left\{ \varphi^i \in \Phi(LB_{P_i}) \right\}_{i=1}^n, \left\{ \varphi_i(LB_{P_i}^m) \right\}_{i=1}^n \right\},$$

where  $m$  denotes the step where the agreement is reached, and  $\infty$  otherwise.

It can be easily checked that the *Diminishing Claims* and the *Unanimous Concessions* procedures are dual, since the maximum amount that each agent can receive in the former mechanism can be interpreted as the minimal losses in which each agent can incur applying the latter mechanism. That is,  $Lc_i(LB_{P_1}) = c_i - Ls_i\left((LB_{P_1})^d\right)$ .

In this sense, Chun [8] shows that in all non-cooperative Nash equilibria induced by *Diminishing Claims* procedure for  $P_1$ , each agent will receive the awards recommended by the *Constrained Equal Awards* rule.

**Theorem 10** (Chun, 1989) *In any Nash equilibrium induced by the game  $\Gamma_{LB_{P_1}}^d$ , each agent receives the amount given by the CEA rule.*

Moreover, by duality, we can retrieve the *Piniles'* rule in  $P_2$ , and the *Constrained Egalitarian* rule for the bi-personal problems but not for the general case in  $P_3$ , when using the *Diminishing Claims* mechanism.

**Corollary 8** *In any Nash equilibrium induced by the game  $\Gamma_{LB_{P_2}}^d$ , each agent receives the amount given by Pin rule.*

**Corollary 9** *For the bi-personal problems  $LB_{P_3} \in \mathcal{LB}_P$ ,  $LB_{P_3} = ((E, c), CE, DCE)$ , in any Nash equilibrium induced by the game  $\Gamma_{LB_{P_3}}^d$ , each agent receives the amount given by the CE rule.*

**Corollary 10** *There is a problem,  $(E, c) \in \mathcal{B}$ , for which in a Nash equilibrium induced by the game  $\Gamma_{LB_{P_3}}^d$ , not all agents receive the amount given by the CE rule.*

#### 4.4 The Double Concessions procedure.

In this section we define a new mechanism which combines the philosophy of the *Diminishing Claims* (Chun [8]) and the *Unanimous Concessions* (Herrero [21]) procedures, using the fact that they are dual.

This new method, named the *Double Concessions* procedure, says that given that agents have chosen their preferred rules, if at the initial step there is no agreement, at the second step, each agent receives the smallest amount among all the proposed at step 1. Now, we redefine the residual bankruptcy problem, in which the endowment is the leftover resources, and the claims are truncated by the maximum amount recommended by all the suggested rules and adjusted down by the amounts just given. Then, the procedure is again applied until an agreement is reached. If this is not the case, the solution will be the limit of the procedure if is feasible, and zero otherwise. Formally:

**Definition 14** *Double Concessions procedure, du:*

Let  $LB_{P_i} \in \mathcal{LB}_{P_i}$ . At the first stage, each agent chooses a rule  $\varphi^i \in \Phi(LB_{P_i})$ . Let  $\psi = (\varphi^i)$  be the profile of selected rules. The proposal of the *Double Concessions* procedure,  $du[\psi, LB_{P_i}]$  is obtained as follows:

[Step 1] If all agents agree on  $\varphi(LB_{P_i})$ , then  $du[\psi, LB_{P_i}] = \varphi LB_{P_i}$ . Otherwise, go to next step.

[Step 2] Let us define

$$Ls_i(LB_{P_i}) = \min_{j \in N} \varphi_i^j(LB_{P_i}),$$

$$Lc_i(LB_{P_i}) = \max_{j \in N} \varphi_i^j(LB_{P_i}),$$

$$c^2 = Lc(LB_{P_i}) - Ls(LB_{P_i}),$$

$$E^2 = E - \sum_{i \in N} Ls_i(LB_{P_i}), \text{ and}$$

$$LB_{P_i}^2 = ((E^2, c^2), LGM^{P_i}, LLM^{P_i}).$$

Now, if all agents agree on  $\varphi(LB_{P_i}^2)$ , then  $du[\psi, LB_{P_i}] = Ls(LB_{P_i}) + \varphi(LB_{P_i}^2)$ .

Otherwise, go to next step.

[Step  $m + 1$ ] Let us define

$$Ls_i(LB_{P_i}^m) = \min_{j \in N} \varphi_i^j(LB_{P_i}^m),$$

$$Lc_i(LB_{P_i}^m) = \max_{j \in N} \varphi_i^j(LB_{P_i}^m),$$

$$c^{m+1} = Lc(LB_{P_l}^m) - Ls(LB_{P_l}^m),$$

$$E^{m+1} = E^m - \sum_{i \in N} Ls_i(LB_{P_l}^m), \text{ and}$$

$$LB_{P_l}^{m+1} = ((E^{m+1}, c^{m+1}), LGM^{P_l}, LLM^{P_l}).$$

Now, if all agents agree on  $\varphi(LB_{P_l}^{m+1})$ , then  $du[\psi, LB_{P_l}] = \sum_{k=1}^m Ls(LB_{P_l}^k) + \varphi(LB_{P_l}^{m+1})$ . Otherwise, go to next step.

[Limit case] Compute  $\lim_{m \rightarrow \infty} \sum_{k=1}^m Ls(LB_{P_l}^k)$ . If it converges to an allocation,  $x$ , such that  $\sum_{i \in N} x_i \leq E$ ,  $du[\psi, LB_{P_l}] = x$ . Otherwise,  $du[\psi, LB_{P_l}] = 0$ .

From now on, let  $\Gamma_{LB_{P_l}}^{du}$  denote the game induced by the *Double Concessions* procedure when agents act strategically, in which the set of players is  $N$  and the strategies for each agent are rules in  $\Phi(LB_{P_l})$  and the payoffs are the sum of the amounts received by each agent in each step  $m \in \mathbb{N}$ . That is,

$$\Gamma_{LB_{P_l}}^{du} = \left\{ N, \left\{ \varphi^i \in \Phi(LB_{P_l}) \right\}_{i=1}^n, \left\{ \sum_{k=1}^m Ls_i(LB_{P_l}^k) \right\}_{i=1}^n \right\},$$

where  $m$  denotes the step where the agreement is reached, and  $\infty$  otherwise.

Next theorem shows the main result when applying the *Double Concessions* procedure in  $P_l$ .

**Theorem 11** *In any Nash equilibrium induced by the game  $\Gamma_{LB_{P_l}}^{du}$ , such that  $P_l$  is Self-Dual, each agent receives the amount given by the average of the two Lorenz-Focal rules.*

**Proof.** See Section C.8 in Appendix C. ■

We know (Section 3.3, Chapter 3) the *Lorenz P-Safety* and the *Lorenz P-Ceiling* for  $P_l \in \{P_1, P_2, P_3\}$ . So next results are straightforwardly obtained by applying Theorem 11.

**Corollary 11** *In any Nash equilibrium induced by the game  $\Gamma_{LB_{P_1}}^{\text{du}}$ , each agent receives the amount given by the average of the Constrained Equal Awards and the Constrained Equal Losses rules.*

**Corollary 12** *In any Nash equilibrium induced by the game  $\Gamma_{LB_{P_2}}^{\text{du}}$ , each agent receives the amount given by the average of Piniles' and the Dual of Piniles' rules.*

**Corollary 13** *In any Nash equilibrium induced by the game  $\Gamma_{LB_{P_3}}^{\text{du}}$ , each agent receives the amount given by the average of the Constrained Egalitarian and the Dual Constrained Egalitarian rules.*

As pointed out in Chapter 3, the convex combination of rules preserves *Resource Monotonicity*, *Super-Modularity* and the *Midpoint* properties (Thomson and Yeh [54]). Hence, as these corollaries show, our new procedure provides *Lorenz-Bifocal Admissible rules* for  $P_l \in \{P_1, P_2, P_3\}$ , while applying independently the *Diminishing Claims* and the *Unanimous Concessions* they fail *Resource Monotonicity* for  $P_3$ .

#### 4.5 Conclusions.

In this chapter we have offered new basis for old bankruptcy rules from a new angle. Specifically, we particularize the methodology of the *Unanimous Concessions procedure* to different sets of "*Commonly Accepted Equity Principles*" by a society.

On the one hand, we have obtained the *DPin* rule when applying the *Unanimous Concessions* procedure for the set  $P_2$ . However, this result cannot be generalized to any set of equity principles  $P_l$ , as we have shown for  $P_3$ .

By using the idea of duality, the previous results can be analyzed from the viewpoint of sharing losses, i.e., we focus on the maximum awards that each agent can ensure. In this case, *Pin* rule is retrieved for  $P_2$  when using the *Diminishing Claims* mechanism.

Finally, we have justified the average of rules by defining a new mechanism, named the *Double Concessions* procedure, combining the *Diminishing Claims* and the *Unanimous Concessions* procedures.



## CHAPTER 5

## LORENZ-BIFOCAL BANKRUPTCY GAME.

## 5.1 Introduction.

This chapter provides a consideration of *Lorenz-Bifocal Bankruptcy Problems* from a Cooperative Game viewpoint. Firstly, these kinds of problems are modelled as transferable utility cooperative games, known as *Lorenz-Bifocal Bankruptcy Games*, by associating each coalition with the smallest quantity of the ‘good’ that such a coalition would receive according to the two proposed allocations. A necessary condition is then provided in order to ensure that a proposal is in the *Core* of these games: an amount belonging to the interval within the range of the two extremes defined by the focal proposals should be recommended for each agent. Furthermore, although these games are not convex in general, the *Shapley value* is observed not only to be a *Core* selection, but also to coincide with the *Nucleolus*. The recommendations made by these two solution concepts are also shown to be the ‘average of the two *Lorenz-Focal rules*’.

The chapter is organized as follows. Section 5.2 formally introduces *Lorenz-Bifocal Bankruptcy Games*, and provides the main characteristics. Section 5.3 applies previous ideas to the three different sets of equity principles. Section 5.4 summarizes our conclusions. Finally, Appendix D contains our technical proofs.

## 5.2 The model.

In this section we define a new type of games in coalitional form, named *Lorenz-Bifocal Bankruptcy Games*, which are an appropriate interpretation of *Lorenz-Bifocal Bankruptcy Problems* in terms of TU-games. Specifically, following the line initiated in Chapter 2, we define the game corresponding to these problems by associating to each coalition the smallest quantity of the resource that it would receive according to all the *Lorenz-Bifocal Admissible rules*. Thus, we maintain the pessimistic viewpoint used by O'Neill [42] to introduce his bankruptcy game, assuming that each group of agents think about their worst situation, i.e., the worst distribution of the endowment for them according to the two *Lorenz-Focal rules*. Formally:

**Definition 15** Given  $LB_{P_i} \in \mathcal{LB}_{P_i}$ , the corresponding **Lorenz-Bifocal Bankruptcy Game** is the TU-game  $V^{LB_{P_i}}$  which associates the real value  $V^{LB_{P_i}}(\emptyset) = 0$  and, for each coalition  $S$ ,  $\emptyset \neq S \subseteq N$ ,

$$V^{LB_{P_i}}(S) = \min \left\{ \sum_{i \in S} LGM_i^{P_i}(E, c), \sum_{i \in S} LLM_i^{P_i}(E, c) \right\}.$$

In order to illustrate our model, we present the following example.

**Example 1** Let us consider the *Lorenz-Bifocal Bankruptcy Problem*  $LB_{P_i} \in \mathcal{LB}_{P_i}$ , in which  $E = 90$ ,  $LGM^{P_i}(E, c) = (30, 30, 30)$  and  $LLM^{P_i}(E, c) = (20, 30, 40)$ . Then, the worth of each coalition is  $V^{LB_{P_i}}(\{1\}) = \min\{30; 20\} = 20$ ;  $V^{LB_{P_i}}(\{2\}) = \min\{30; 30\} = 30$ ;  $V^{LB_{P_i}}(\{3\}) = \min\{30; 40\} = 30$ ;  $V^{LB_{P_i}}(\{1, 2\}) = \min\{30 + 30; 20 + 30\} = 50$ ;  $V^{LB_{P_i}}(\{1, 3\}) = \min\{30 + 30; 20 + 40\} = 60$ ;  $V^{LB_{P_i}}(\{2, 3\}) = \min\{30 + 40; 30 + 30\} = 60$ ;  $V^{LB_{P_i}}(\{1, 2, 3\}) = 90$ .

Next, we analyze the basics properties and the main solution concepts of *Lorenz-Bifocal Bankruptcy Games*.

Note that a *Lorenz-Bifocal Bankruptcy Game*,  $V^{LB_{P_i}}$  with  $LB_{P_i} \in \mathcal{LB}_P$ , has a non-empty *Core*, since both  $LGM^{P_i}(E, c)$  and  $LLM^{P_i}(E, c)$  belong to it. Moreover, next proposition provides a set of necessary conditions for a proposal to be in the *Core* of a such game, which coincides with the "natural" requirements for a rule to be considered a *Lorenz-Bifocal Admissible rule* (see Definition 4). This result, therefore, shows that the agents' behavior regarding *Lorenz-Bifocal Bankruptcy Problems* has, albeit unconsciously, strong theoretical support.

**Proposition 8** *Given  $LB_{P_i} \in \mathcal{LB}_P$ , if  $x \in \mathbb{C}(V^{LB_{P_i}})$  then, for all  $i \in N$ ,*

$$\min\{LGM_i^{P_i}(E, c), LLM_i^{P_i}(E, c)\} \leq x_i \leq \max\{LGM_i^{P_i}(E, c), LLM_i^{P_i}(E, c)\}.$$

**Proof.** See Section D.1 in Appendix D. ■

However, the following example shows that previous conditions do not always guarantee that an allocation belongs to the *Core*. That is, it is not a sufficient requirement.

**Example 2** *Let us consider the Lorenz-Bifocal Bankruptcy Problem  $LB_{P_i} \in \mathcal{LB}_P$ , in which  $E = 170$ ,  $LGM^{P_i}(E, c) = (35, 45, 45, 45)$  and  $LLM^{P_i} = (28.75, 38.75, 48.75, 53.75)$ . Then, on the one hand  $V^{LB_{P_i}}(\{1, 3\}) = \min\{35 + 45; 28.75 + 48.75\} = 77.5$ . On the other hand, let us consider the allocation  $x$  that distributes the resource,  $E = 170$ , among agents as follows,  $x = (30, 40.25, 46, 53.75)$ . This allocation is, for each agent, between those of the two Lorenz-Bifocal rules. However,*

$$x_1 + x_3 = 76 < V^{LB_{P_i}}(\{1, 3\}) = 77.5,$$

*therefore  $x$  does not belongs to  $\mathbb{C}(V^{LB_{P_i}})$ .*

It is well known that the *Nucleolus* is a *Core* selection. Nevertheless, the *Shapley value* for convex games it is also a *Core* selection, it could recommend, in general, a

distribution outside the *Core*. In this line, the following example shows that *Lorenz-Bifocal Bankruptcy Games* need not be convex.

**Example 3** Let us consider the *Lorenz-Bifocal Bankruptcy Problem*  $LB_{P_i} \in \mathcal{LB}_P$ , in which  $E = 30$ ,  $LGM^{P_i}(E, c) = (7.5, 7.5, 7.5, 7.5)$  and  $LLM^{P_i}(E, c) = (5, 7, 8, 10)$ . It can be easily verified that  $V^{LB_{P_i}}(\{1\}) = 5$ ,  $V^{LB_{P_i}}(\{1, 3\}) = 13$ ,  $V^{LB_{P_i}}(\{1, 4\}) = 15$  and  $V^{LB_{P_i}}(\{1, 3, 4\}) = 22.5$ . Thus,  $\Delta_3 V^{LB_{P_i}}(\{1, 4\}) < \Delta_3 V^{LB_{P_i}}(\{1\})$  since

$$V^{LB_{P_i}}(\{1, 3, 4\}) - V^{LB_{P_i}}(\{1, 4\}) = 7.5 \text{ and } V^{LB_{P_i}}(\{1, 3\}) - V^{LB_{P_i}}(\{1\}) = 8.$$

Therefore, given that  $\{1\} \subset \{1, 4\}$ ,  $V^{LB_{P_i}}$  is not convex.

So, in order to choose a *Core* selection, the above example lead us to focus on the *Nucleolus* of the game. However, as we will show in our main result (Theorem 12), the *Shapley value* belongs to the *Core* and coincides with the *Nucleolus*. The reason of this coincidence lies in the fact that in our game the sum of a player's marginal contribution to any pair of disjoint coalitions  $T, T^*$  such that  $T \cup T^* = N \setminus \{i\}$  is a player specific constant, i.e., the *Lorenz-Bifocal Bankruptcy Game* is a *PS-game* (Kar et al. [25]). Formally:

A TU-game  $V$  is a **PS-game** if for each  $i \in N$ , there exists  $k_i \in \mathbb{R}$  such that, for all  $T \subseteq N \setminus \{i\}$ ,

$$\Delta_i V(T) + \Delta_i V(N \setminus [T \cup \{i\}]) = k_i.$$

The following result shows that *Lorenz-Bifocal Bankruptcy Games* are a subclass of *PS-games*. Moreover, the specific constant, for each agent, obtained by adding its marginal contribution to any coalition  $S$  and its complement  $N \setminus [S \cup \{i\}]$ , is the sum of the recommendations for her made by the two *Lorenz-Focal rules*.

**Proposition 9** Given  $LB_{P_i} \in \mathcal{LB}_{P_i}$ , the associated *Lorenz-Bifocal Bankruptcy Game*,  $V^{LB_{P_i}}$ , is a PS-game such that for all  $i \in N$  and for all coalition  $T \subseteq N \setminus \{i\}$ ,

$$\Delta_i V^{LB_{P_i}}(T) + \Delta_i V^{LB_{P_i}}(N \setminus [T \cup \{i\}]) = LGM_i^{P_i}(E, c) + LLM_i^{P_i}(E, c).$$

**Proof.** See Section D.2 in Appendix D. ■

Our main result, the proof of which is based on states with sufficient clarity how to solve *Lorenz-Bifocal Bankruptcy Games*. In a way, it provides solid grounds for selecting the average of the two focal viewpoints from among all the intermediate compromises: "*Virtue lies in the middle ground*".

**Theorem 12** For each *Lorenz-Bifocal Bankruptcy Game*,  $V^{LB_{P_i}}$ , where  $LB_{P_i} \in \mathcal{LB}$ , the Shapley value and the Nucleolus coincide and they are obtained in the average of the two *Lorenz-Focal rules*, that is,

$$Sh(V^{LB_{P_i}}) = Nu(V^{LB_{P_i}}) = 1/2 (LGM^{P_i}(E, c) + LLM^{P_i}(E, c)).$$

**Proof.** See Section D.3 in Appendix D. ■

### 5.3 Three applications.

In this section we apply the proposed model to the three different sets of equity principles mentioned previously. Particularly, from Chapter 2, we know the *Lorenz-Focal rules* which delimit the region of *Lorenz-Bifocal Admissible rules* are well defined with  $P_i \in \{P_1, P_2, P_3\}$ . Therefore, by Definition 15 we obtain each of the associated *Lorenz-Bifocal Bankruptcy Games*.

Given  $LB_{P_i} \in \mathcal{LB}_{P_i}$ ,  $LB_{P_i} = ((E, c), CEA, CEL)$ , the corresponding **Lorenz-Bifocal Bankruptcy Game** is the TU-game  $V^{LB_{P_i}}$  which associates to each coalition  $S \subseteq N$ , the real value

$$V^{LB_{P_i}}(S) = \min \left\{ \sum_{i \in S} CEA_i(E, c), \sum_{i \in S} CEL_i(E, c) \right\}.$$

Given  $LB_{P_2} \in \mathcal{LB}_P$ ,  $LB_{P_2} = ((E, c), Pin, DPin)$ , the corresponding **Lorenz-Bifocal Bankruptcy Game** is the TU-game  $V^{LB_{P_2}}$  which associates to each coalition  $S \subseteq N$ , the real value

$$V^{LB_{P_2}}(S) = \min \left\{ \sum_{i \in S} Pin_i(E, c), \sum_{i \in S} DPin_i(E, c) \right\}.$$

Given  $LB_{P_3} \in \mathcal{LB}_P$ ,  $LB_{P_3} = ((E, c), CE, DCE)$ , the corresponding **Lorenz-Bifocal Bankruptcy Game** is the TU-game  $V^{LB_{P_3}}$  which associates to each coalition  $S \subseteq N$ , the real value

$$V^{LB_{P_3}}(S) = \min \left\{ \sum_{i \in S} CE_i(E, c), \sum_{i \in S} DCE_i(E, c) \right\}.$$

Obviously, in each of these *Lorenz-Bifocal Bankruptcy Games* all the characteristics and results set out in the previous section hold. Then, by applying Theorem 12 for each  $P_i \in \{P_1, P_2, P_3\}$ , we present a game-theoretic support of new rules: the average of well-known old solutions representing opposite viewpoints.

**Corollary 14** *For each Lorenz-Bifocal Bankruptcy Game,  $V^{LB_{P_1}}$ , the Shapley value and the Nucleolus coincide. They are also defined by the average of the Constrained Equal Awards and the Constrained Equal Losses rules, that is,*

$$Sh(V^{LB_{P_1}}) = Nu(V^{LB_{P_1}}) = 1/2 [CEA(E, c) + CEL(E, c)].$$

**Corollary 15** *For each Lorenz-Bifocal Bankruptcy Game,  $V^{LB_{P_2}}$ , the Shapley value and the Nucleolus coincide. They are defined by the Average of Piniles' and the Dual of Piniles' rules, that is,*

$$Sh(V^{LB_{P_2}}) = Nu(V^{LB_{P_2}}) = 1/2 [Pin(E, c) + DPin(E, c)].$$

**Corollary 16** *For each Lorenz-Bifocal Bankruptcy Game,  $V^{LB_{P_3}}$ , the Shapley value and the Nucleolus coincide. They are defined by the Average of the Constrained Egalitarian and the Dual Constrained Egalitarian rules, that is,*

$$Sh(V^{LB_{P_3}}) = Nu(V^{LB_{P_3}}) = 1/2 [CE(E, c) + DCE(E, c)].$$

At this point it should be noted that the compromise reached by the society that we have considered could be classed as coherent, at least, for two reasons. The first of these is that the properties that the society has applied to restrict the set of *Lorenz-Bifocal Admissible rules* in each case,  $P_i \in \{P_1, P_2, P_3\}$ , are also satisfied by the final agreement. The second reason is that the "equity criterion" on which the society bases its choices, *Lorenz domination*, is preserved, although the application of such a criterion is carried out in different domains, each one corresponding to the problem facing society at each time (allocations for bankruptcy problems and excesses of coalitions for bankruptcy games). In this regard, it is well known that, for each TU-game  $V$ , the *Nucleolus* selects the allocation of the resource providing the maximal Lorenz element on the set of vectors the coordinates of which are the excesses of all coalitions referring to efficient and individually rational distributions,  $x = V(N)$  and  $x_i \geq V(\{i\})$  for all  $i \in N$ , (see Arin [4]).

#### 5.4 Conclusions.

In this chapter we have used cooperative game theory to support very simple, current and commonly observed collective decisions, which are so common as to be expressed in the popular proverb: "*Virtue lies in the middle ground*".

Our main result offers game-theoretic grounds for meeting people half-way, providing solid grounds for selecting the average of the two focal viewpoints from among all the *Lorenz-Bifocal Admissible rules*.

On the other hand, we have particularized this model to the three sets of equity principles proposed, showing how to introduce new applications of game theory and, consequently, new solution concepts by following the line of argument of Section 5.3.

Therefore, we can conclude that this chapter involves both simplicity and the matching of theory with real life, producing an interesting combination.

## CHAPTER 6

## GENERAL CONCLUSIONS AND FUTURE RESEARCH.

## 6.1 Results.

Our contribution to the theory about bankruptcy situations in this document starts from two general concerns, which are usual in the literature: individual rights and fair distributions. Regarding to these ideas, we introduce a new extended bankruptcy problem, called the *Lorenz-Bifocal Bankruptcy Problem*, in which the discrepancy for sharing the resources is considered by means of the existence of two focal rules, according to a "*Commonly Accepted Equity Principles*" set.

Taking these problems as starting point, we have analyzed the repercussions of this enrichment of the classical rationing problem from the Axiomatic, Strategic and Cooperative Game viewpoints, obtaining the following results:

- In the Axiomatic Approach, firstly, we provide the definition of a lower bound, called *Lorenz P-Safety*, which is obtained by assigning to each agent the smallest amount she receives according to all *Lorenz-Bifocal Admissible rules* for such a society. Secondly, the fact that some of the endowment will be still available once we allocate each agent this amount, has led us to introduce the *Recursive Lorenz P-Safety* rule, defined as the recursive application of our new bound. In this sense, our main results are achieved by particularizing the previous methodology to different sets of equity principles which can be interpreted, from our point of view, as "basic" requirements. For two

possible societies (restrictive and permissive) we retrieve, respectively, the *Dual of Piniles'* and the *Constrained Equal Losses* rules when focusing on awards; and *Piniles'* and the *Constrained Equal Awards* rules when focusing on losses. Nevertheless, by considering a society whose "*Commonly Accepted Equity Principles*" set is included in the permissive society's set and includes the restrictive society's set, we show that the *Recursive Lorenz P-Safety* rule does not satisfy one of the equity principles upon which such a society initially agreed to found its decisions. Thus, the need for in-depth study of the consequences of social agreements on both principles and procedure has been emphasized, as they could become meaningless when combined.

In this line, and using the fact that rationing problems can be interpreted from a double viewpoint (awards and losses), we propose the simultaneous combination of both a lower and an upper bound on awards, called the *Lorenz P-Safety* and the *Lorenz P-Ceiling*, respectively. In this regard, we define a new method for distributing the endowment, using the idea of establishing recursively a "fair" minimum and maximum amounts to each agent in bankruptcy problems, named the *Lorenz Double Recursive rule*. Our main result states that this rule corresponds with the average of the two *Lorenz-Focal rules*, that is, we obtain new justifications for the convex combination of two extreme and opposite rules. Then, we particularize this procedure to some sets of equity principles, recovering *Lorenz-Bifocal Admissible rules* even for the set of equity principles for which the *Recursive Lorenz P-Safety* process fails.

- From the Strategic Approach we provide a mechanism from a positive viewpoint. That is, we offer the understanding of old bankruptcy rules as the *Nash equilibrium* of a strategic game where agents act individually pursuing

their best strategy, the one ensuring her highest payoff. Specifically, we particularize the methodology of the *Unanimous Concessions* (Marco et al. [31]) and the *Diminishing Claims* (Chun [8]) procedures to different sets of "Commonly Accepted Equity Principles". In particular, we retrieve the *Dual of Piniles'* rule when applying the *Unanimous Concessions* procedure, and, by using the idea of duality and the fact all the proposed properties are *Self-Dual*, we obtain *Piniles'* rule. Moreover, we show that, for some set of equity principles, the allocation obtained when applying the *Unanimous Concessions* and the *Diminishing Claims* procedures, may lead to "not acceptable" results. Particularly, if a society agreed on choosing those rules which satisfy a determined set of equity principles, the final allocation could not satisfy the initial agreed properties.

At this point, combining the *Diminishing Claims* and the *Unanimous Concessions* procedures, we define the *Double Concessions* procedure. In this case, as in the Axiomatic Approach, we retrieve the average of the two *Lorenz-Focal* rules for the three proposed sets of properties.

- In the Cooperative Game Approach, we associate to our new extended bankruptcy problem a natural TU game, applying the O'Neill's model [42]. In these games, called *Lorenz-Bifocal Bankruptcy Games*, we obtain necessary conditions for solutions to be a *Core* selection. On the other hand, although these games are not convex in general, we find that the *Shapley Value* is not only in the *Core*, but it also coincides with the *Nucleolus*. Moreover, these two solutions retrieve the average between the two *Lorenz-Focal rules*, that is, we provide game theoretical grounds to sustain the proposal of an "intermediate" agreement among all the *Lorenz-Bifocal Admissible rules*.

## 6.2 Future research in progress.

From our viewpoint, there are several researching possibilities following the line initiated in the current dissertation. Among other problems, at this moment we are facing two possible generalizations of *Lorenz-Bifocal Bankruptcy Problems*. On the one hand, note that, by applying the *Lorenz criterion*, we have obtained two focal rules which delimit the *Lorenz-Bifocal Admissible rules*. So that, a natural step for trying to generalize our results would be by "substracting" these extreme rules not requiring the equity criterion. Particularly, we relax the requirements imposed on the initial extended bankruptcy problem not considering the *Lorenz criterion*. On the other hand, we consider other contexts, different of bankruptcy problems, where the two focal proposals arises in a natural way.

### 6.2.1 Bankruptcy Problems with Legitimate Principles.

Our first attempt to extend the obtained results consists of eliminating the *Lorenz criterion*, and then considering that the way of distributing the resource is only based on a set of basic properties, called "*Commonly Accepted Equity Principles*", but without requiring as social goal an equity criterion. Formally:

**Definition 16** *A Bankruptcy Problem with Legitimate Principles is a triplet  $(E, c, P_l)$ , where  $(E, c) \in \mathcal{B}$  and  $P_l$  is a fixed set of principles on which a particular society has agreed.*

Let  $P$  be the set of all subsets of properties of rules. Each  $P_l \in P$  represents a specific society which will always apply such principles for solving its problems. Finally, let  $\mathcal{B}_P$  be the set of all *Problems with Legitimate Principles*.

In this context, a *Socially Admissible* rule for a society that has agreed on  $P_l$ , is a rule satisfying all these properties.

**Definition 17** A *Socially Admissible rule*, or simply an *Admissible rule*, is a function,  $\varphi : \mathcal{B}_P \rightarrow \mathbb{R}_+^n$ , such that its restriction on  $\mathcal{B}$ ,  $\varphi : \mathcal{B} \rightarrow \mathbb{R}_+^n$ , is a rule, and it satisfies all properties in  $P_l$ .

Let  $\Phi$  denote the set of all rules and let  $\Phi(P_l)$  be the subset of rules satisfying  $P_l$ .

### Bankruptcy Problems with Legitimate Principles: Lower bound.

- **Axiomatic Approach.**

Now, from the Axiomatic Approach, and following a similar reasoning as in Chapter 2, we define the lower bound on awards based on *Bankruptcy Problems with Legitimate Principles* and its associated recursive process.

**Definition 18** Given  $(E, c, P_l)$  in  $\mathcal{B}_P$ , the ***P-Safety***,  $s$ , is for each  $i \in N$ ,

$$s_i(E, c, P_l) = \inf_{\varphi \in \Phi(P_l)} \{\varphi_i(E, c)\}.$$

**Definition 19** For each  $m \in \mathbb{N}$ , the ***Recursive P-Safety Process*** at the  $m$ -th step,  $RS$ , associates for each  $(E, c, P_l) \in \mathcal{B}_P$  and each  $i \in N$ ,

$$[RS(E^m, c^m, P_l)]_i = s_i(E^m, c^m, P_l),$$

where  $(E^1, c^1) \equiv (E, c)$  and for  $m \geq 2$ ,

$$(E^m, c^m) \equiv (E^{m-1} - \sum_{i \in N} s_i(E^{m-1}, c^{m-1}, P_l), c^{m-1} - s(E^{m-1}, c^{m-1}, P_l)).$$

Again, it can not be ensured that the sum of the amounts that agents receive in each step exhausts the endowment, but when this occurs, we call this sum the *Recursive P-Safety* rule. Moreover, note that this rule will be an *Admissible* rule whenever it fulfills the set of equity principles on which the *Recursive P-Safety Process* is based. A fact that, as we will see later, cannot always be guaranteed.

**Definition 20** The *Recursive P-Safety* rule,  $\varphi^R$ , associates for each  $(E, c, P_l) \in \mathcal{B}_P$  and each  $i \in N$ ,  $\varphi_i^R(E, c, P_l) = \sum_{m=1}^{\infty} [RS(E^m, c^m, P_l)]_i$ , whenever

$$\sum_{i \in N} \left( \sum_{m=1}^{\infty} [RS(E^m, c^m, P_l)]_i \right) = E.$$

At this point, we consider the three possible choices of "Commonly Accepted Equity Principles" by a society to apply the approach introduced previously,  $P_l \in \{P_1, P_2, P_3\}$ , although with some conditions. Particularly, now we include *Order Preservation* and *Resource Monotonicity*.

**Order Preservation:** for each  $(E, c) \in \mathcal{B}$ , and each  $i, j \in N$ , such that  $c_i \geq c_j$ , then  $\varphi_i(E, c) \geq \varphi_j(E, c)$ , and  $c_i - \varphi_i(E, c) \geq c_j - \varphi_j(E, c)$ , that is  $l_i(E, c) \geq l_j(E, c)$ .

**Resource Monotonicity:** for each  $(E, c) \in \mathcal{B}$  and each  $E' \in \mathbb{R}_+$  such that  $C > E' > E$ , then  $\varphi_i(E', c) \geq \varphi_i(E, c)$ , for each  $i \in N$ .

So our sets of equity principles are:

$P_1 = \{\text{Efficiency, Claim-boundedness, Non-Negativity, Order Preservation and Resource Monotonicity}\}$ ,

$P_2 = \{\text{Efficiency, Claim-boundedness, Non-Negativity, Resource Monotonicity, Super-Modularity and Midpoint Property}\}$ ,

and

$P_3 = \{\text{Efficiency, Claim-boundedness, Non-Negativity, Order Preservation, Resource Monotonicity and Midpoint Property}\}$ .

It must be remarked that, in the context of *Lorenz-Bifocal Bankruptcy Problems*, these new conditions are redundant. So, in the context of the *Bankruptcy Problems*

with *Legitimate Principles*, the sets of equity principles coincide with the ones we are presenting now.

By applying our previous methodology, we now recover the *Constrained Equal Losses* and the *Dual of Piniles'* rules for  $P_1$  and  $P_2$ . However, although for two-person problems the *Recursive P-Safety* rule for  $P_3$  retrieves the *Dual Constrained Egalitarian* rule, this fact can not be extended for the n-person case.

**Proposition 10** For each  $(E, c, P_1) \in B_P$ ,  $((E, c, P_2) \in B_P)$ ,  $((E, c, P_3) \in B_P)$ , the *P-Safety* for the smallest agent, say 1, corresponds with the *Constrained Equal Losses (the Dual of Piniles')* (*the Dual Constrained Egalitarian*) rule,

$$s_1(E, c, P_1) = CEL_1(E, c) \quad (s_1(E, c, P_2) = DPin_1(E, c)) \quad (s_1(E, c, P_3) = DCE_1(E, c)).$$

**Proof.** See Appendix E. ■

Then, by applying duality, it is straightforwardly to show the *P-Safety* for the highest agent.

**Corollary 17** For each  $(E, c, P_1) \in B_P$ ,  $((E, c, P_2) \in B_P)$ ,  $((E, c, P_3) \in B_P)$ , the *P-Safety* for the highest agent, say  $n$ , corresponds with the *Constrained Equal Awards (Piniles')* (*Constrained Egalitarian*) rule,

$$s_n(E, c, P_1) = CEA_n(E, c) \quad (s_n(E, c, P_2) = Pin_n(E, c)) \quad (s_n(E, c, P_3) = CE_n(E, c)).$$

The most important results that we have obtained are the following ones:

**Theorem 13** For each  $(E, c, P_1) \in B_P$ ,  $((E, c, P_2) \in B_P)$ , the *Recursive P-Safety* rule is the *Constrained Equal Losses (the Dual of Piniles')* rule,

$$\varphi^R(E, c, P_1) = CEL(E, c) \quad (\varphi^R(E, c, P_2) = DPin(E, c)).$$

**Theorem 14** For each bi-personal Bankruptcy Problem with *Legitimate Principles* in  $B_P$  with  $P = P_3$  and each  $i \in \{1, 2\}$ , the *Recursive P-Safety* rule is the *Dual Constrained Egalitarian* rule,  $\varphi_i^R(E, c, P_3) = DCE_i(E, c)$ .

**Proposition 11** *There is a problem,  $(E, c) \in \mathcal{B}$ , for which the Recursive  $P$ -Safety rule for  $P_3$  does not coincide with the Dual Constrained Egalitarian rule,*

$$\varphi^R(E, c, P_3) \neq DCE(E, c).$$

**Proposition 12** *For  $P_3$ , the Recursive  $P$ -Safety rule does not satisfy Resource Monotonicity.*

The previous analysis can be rewritten for losses by using the idea of duality. As we know, when focusing on losses, the starting point will be the same sets of "Commonly Accepted Equity Principles",  $P_1$  and  $P_2$ , since all the considered properties are *Self-Dual*. Furthermore, by defining the  $P$ -Safety for the associated problem  $(L, c)$  for each  $(E, c) \in \mathcal{B}$  and applying it recursively, it can be shown that *Piniles'* and the *Constrained Equal Awards* rules are retrieved for  $P_1$  and  $P_2$ , respectively. And, again, only for bi-personal problems for  $P_3$  we retrieve the *Constrained Egalitarian* rule. However, this fact can not be extended for the  $n$ -person case.

As a consequence of Proposition 10 and Corollary 17, it would be worth to note that for bi-personal problems, the two contexts (*Lorenz-Bifocal Bankruptcy Problems*, and *Bankruptcy Problems with Legitimate Principles*) coincide, since the properties gathered by the sets  $P_1$ ,  $P_2$  and  $P_3$ , imply the existence of the two extreme rules defined in each case. Moreover, although the smallest agent's  $P$ -Safety for  $P_1$  ( $P_2$ ,  $P_3$ ) corresponds with the *Constrained Equal Losses* (*Dual of Piniles'*, *Dual Constrained Egalitarian*) rules and the highest agent's  $P$ -Safety for  $P_1$  ( $P_2$ ,  $P_3$ ) corresponds with the *Constrained Equal Awards* (*Piniles'*, *Constrained Egalitarian*) rules, the other agents'  $P$ -Safety will not be determined by these rules, as shown in Example 4 in Appendix E.

- **Strategic Approach.**

In the Strategic Approach, we can generalize all the results obtained in Chapter 4 to *Bankruptcy Problem with Legitimate Principles*, obtaining analogous conclusions. In this sense, again, note that we can use the proofs without making any changes, because, due to the fact that the *P-Safety* for the smallest and the highest agent do not change (Proposition 10 and Corollary 17), their weakly dominant strategies remain the same.

### Bankruptcy Problems with Legitimate Principles: Lower and upper bound.

- **Axiomatic Approach.**

As we know, bankruptcy problems can be faced from the double viewpoint of awards and losses, so we define an upper bound on awards, named the *P-Ceiling*, as the highest amount assumed for each agent by all the *Admissible rules* in  $P_l$ . Next, we propose the *Double Boundedness Recursive Process* as the procedure in which at each step each agent receives a minimum and a maximum amount according to all the *Admissible rules*, i.e., at each step, every agent's claim is truncated by her *P-Ceiling*, and each of them receives her *P-Safety*.

**Definition 21** Given  $(E, c, P_l)$  in  $\mathcal{B}_P$ , the ***P-Ceiling***,  $ce$ , is for each  $i \in N$ ,

$$ce_i(E, c, P_l) = \sup_{\varphi \in \Phi(P_l)} \{\varphi_i(E, c)\}.$$

**Definition 22** Given  $m \in \mathbb{N}$ , the *Double Boundedness Recursive Process*,  $DBR$ , associates for each  $(E, c, P_l) \in \mathcal{B}_P$  and each  $i \in N$ ,

$$[DBR(E^m, c^m, P_l)]_i = s_i(E^m, c^m, P_l),$$

where  $(E^1, c^1) \equiv (E, c)$  and for  $m \geq 2$ ,

$$E^m \equiv E^{m-1} - \sum_{i \in N} s_i(E^{m-1}, c^{m-1}, P_l).$$

$$c_i^m = ce_i(E^{m-1}, c^{m-1}, P_l) - s_i(E^{m-1}, c^{m-1}, P_l).$$

According to this process, an agent will get at the first step her *P-Safety* of the original problem. At the second step, we redefine a residual problem, in which the endowment is the remaining resources and each agent's claim is truncated by her *P-Ceiling*, and then adjusted down by the amount received. Then each agent receives her *P-Safety* of this residual problem, and so on. We can easily see that this process is not always efficient, but when it does, we call it the *Double Recursive* rule. Formally:

**Definition 23** The *Double Recursive* rule,  $\varphi^{DR}$ , associates for each  $(E, c, P_l) \in \mathcal{B}_P$  and each  $i \in N$ ,  $\varphi_i^{DR}(E, c, P_l) = \sum_{m=1}^{\infty} [DBR(E^m, c^m, P_l)]_i$ , whenever

$$\sum_{i \in N} \left( \sum_{m=1}^{\infty} [DBR(E^m, c^m, P_l)]_i \right) = E.$$

As we have noted previously, due to the fact that for bi-personal problems both contexts (*Lorenz-Bifocal Bankruptcy Problems*, and *Bankruptcy Problems with Legitimate Principles*) coincide, the results obtained in Chapter 3 remain valid. However, this fact does not occur generally, as shown in Example 5 in Appendix E. Moreover, it is worth to highlight that for both the *Recursive P-Safety rule* and the *Double Recursive rule* there are more than two focal rules. Nevertheless, in each context

the behavior of the procedure is different, implying that although in the former case (with only the lower bound, or "simple" bounding) the results obtained in the context of *Lorenz-Bifocal Bankruptcy Problems* are maintained, this does not occur with the "double" bounding.

Finally, it must be noticed that even if there are two focal rules, the *Double Boundedness Recursive* rule for  $P_l$  may not retrieve the average of them if these two rules are not dual each other. Henceforth, for any *Bankruptcy Problem with Legitimate Principles* such that there exist two *Focal* rules, which are dual each other, the final allocation provided by the *Double Recursive* rule will correspond with the average of these two *Focal* rules.

- **Strategic Approach.**

Note that in the Strategic Approach bi-personal problems for both contexts (*Lorenz-Bifocal Bankruptcy Problems*, and *Bankruptcy Problems with Legitimate Principles*) also coincide, therefore, all the results obtained in Chapter 4 remain valid. However, due to the fact that for the n-person case we can find situations with more than two focal rules, this fact cannot be generalized.

Furthermore, as in the previous approach, when there are only two focal rules, we retrieve the average of them only when they are dual each other.

### **Bankruptcy Problems with Legitimate Principles: Cooperative Game Approach.**

Finally, from a Cooperative Game viewpoint, we define a new type of games in coalitional form, named *Bankruptcy Games with Legitimate Principles*, which are an appropriate interpretation of *Bankruptcy Problems with Legitimate Principles*

in terms of TU-games. Specifically, following the line initiated in Chapter 5, we define the game corresponding to these problems by associating to each coalition the smallest quantity of the resource that it would receive according to all the *Admissible* rules. Formally:

**Definition 24** Given  $B_{P_i} \in \mathcal{B}_P$ , the corresponding *Bankruptcy Game with Legitimate Principles* is the TU-game  $V^{B_{P_i}}$  which associates the real value  $V^{B_{P_i}}(\emptyset) = 0$  and, for each coalition  $S$ ,  $\emptyset \neq S \subseteq N$ ,

$$V^{B_{P_i}}(S) = \min_{\varphi \in \Phi(P_i)} \left\{ \sum_{i \in S} \varphi_i(E, c, P_i) \right\}.$$

At this point, note that it cannot be ensured that there are just two extreme *Admissible* rules defining the worth of each coalition, as it occurs with the *Lorenz-Bifocal Bankruptcy Game*. Henceforth, we can find *Bankruptcy Games with Legitimate Principles* with more than two focus. In this case, we show through Example 6 (Appendix E) that neither the *Shapley value* and the *Nucleolus* coincide, nor any of them retrieves the average of the focal rules.

### 6.2.2 Bifocal Distribution Problems: Grounds for the average of two focus.

Since previous results cannot be generalized for problems with more than two focal rules, in order to provide more general results, we propose distribution problems in which two prominent proposals are considered equally fair from different reasonable viewpoints. In this regard,

A **distribution problem with transferable utility**,  $D$ , (referred to hereinafter as a **distribution problem**) is formally described by a pair  $D = (M, C^N)$  where  $M \in \mathbb{R}_{++}$  represents a given amount of a perfectly divisible "good", a valuable resource that should be distributed among the agents in  $N =$

$= \{1, \dots, i, \dots, n\}$ . And  $C^N$  is a set of relevant information, concerning the agents, which should somehow be taken into account when solving the problem. Let  $\mathcal{D}$  denote the type of all distribution problems.

In this general context a **distribution rule** is a function,  $f : \mathcal{D} \rightarrow \mathbb{R}^n$ , which proposes an efficient allocation of the resource for each distribution problem  $D \in \mathcal{D}$ , that is,  $\sum_{i \in N} f_i(D) = M$ . Let  $\mathcal{F}$  be a family of rules.

Note that a wide range of real situations can be modeled in this way, such as certain important types of TU-games: market games (Shapley and Shubik [50]), cost allocation games (Young [62]) and simple games (Shapley [49]).

In this context, a *Bifocal Distribution Problem*, denoted by  $BD \in \mathcal{BD}$ , is a triplet  $BD = (D, f, g)$  where  $D = (M, C^N) \in \mathcal{D}$  and both  $f$  and  $g$  are "fixed" distribution rules representing two focal proposals in a particular society.

- **Cooperative Game Approach.**

Regarding to this framework, we define a *Bifocal Distribution Game* by associating to each coalition the smallest quantity of the resource that would receive according to the two focal rules,

**Definition 25** *Given  $BD \in \mathcal{BD}$ , the corresponding **Bifocal Distribution Game** is the TU-game  $V^{BD}$  which associates the real value  $V^{BD}(\emptyset) = 0$  and, for each coalition  $S$ ,  $\emptyset \neq S \subseteq N$ ,*

$$V^{BD}(S) = \min_{f \in \mathcal{F}} \left\{ \sum_{i \in S} f_i(D), \sum_{i \in S} g_i(D) \right\}.$$

Finally, note that, by only changing the rules considered in *Lorenz-Bifocal Bankruptcy Games*, for the focal distribution rules  $f$  and  $g$ , we can follow the same reasoning. So that, next theorem states that for each *Bifocal Distribution Game*, "*Virtue lies in the middle ground*".

**Theorem 15** *For each Bifocal Distribution Game,  $V^{BD}$ , where  $BD = (D, f, g) \in \mathcal{BD}$ , the Shapley value and the Nucleolus coincide and they are obtained in the average of the two focal distributions rules, that is,*

$$Sh(V^{BD}) = Nu(V^{BD}) = 1/2 (f(D) + g(D)).$$

### 6.2.3 Other open research issues.

Finally, other questions that remain open are the following.

- The axiomatic characterization of the convex combinations of rules. Particularly, following Thomson and Yeh [54], it would be interesting to characterize the average of two extreme and opposite rules.
- The search for new procedures that ensure compatibility with socially accepted equity principles; and the analysis of conditions on the legitimate principle sets which guarantee that such principles are upheld when applying our recursive process.
- Following Luttens [29], it could be analyzed the results obtained by the combination of other lower bounds proposed in the economic literature (different from the O'Neill's *Minimal Right* [42]) with the idea of solidarity. In this regard, Luttens, in a model where individuals with different levels of skills exert different levels of effort, combines the idea of solidarity with the individual's bound as defined by O'Neill [42]. Solidarity is based on the notion that the consequences of a change in an individual's skill should be borne by the society as whole. Specifically, let us suppose a society where agents' income inequalities are determined by unequally exerted effort levels and different innate skills. Now suppose that, due to a change in individuals' skills, there is

more income to divide. The proposal is for these extra resources to be divided according to the changes in individuals' minimal rights.

- As Schokkaert and Overlaet [46] gather, we can find many contexts where two different proposals that highlight the discrepancy about the way of distributing the resources appear in a natural way. For instance, the two focal solutions are, following Moulin [35], in surplus sharing problems, the equal and proportional sharing rules; or, according to Varian [59], in the theory of fairness and envy-free allocation, the income-fair and the wealth-fair allocations. In this sense, on the one hand, we have provided general results using the Cooperative Game Approach for any *Bifocal Distribution Game*. On the other hand, it would be interesting to analyze if we can retrieve similar results as those obtained in the context of *Lorenz-Bifocal Bankruptcy Problems* in the Axiomatic and Strategic Approaches.
- Following Amiel and Cowell [3], and Bosmans and Schokkaert [7], it could be examined public perceptions of "fair", "appealing", or "reasonable" guarantees for rationing problems. Specifically, it would be interesting to answer the following questions. Does the standard structure of analyzing guarantees coincide with its public perceptions? Is the economist's concept of guarantee a thing apart, that is going to be perpetuated through social brainwashing in the way the subject is studied and taught?



## APPENDIX A

## PROOFS OF CHAPTER 2.

## A.1 General remarks.

We present three remarks which are used in the following proofs.

Let us note that  $\mathcal{B}_0$  denotes the set of bankruptcy problems in which claims are increasingly ordered, that is, bankruptcy problems with  $c_i \leq c_j$  for  $i < j$ .

First, for any *Lorenz-Bifocal Bankruptcy Problem*, the total loss to distribute is the same at every step of the *Recursive Lorenz P-Safety Process*.

**Remark 1** For each  $LB_{\rho_i} \in \mathcal{LB}_{\rho}$ , and each  $m \in \mathbb{N}$ ,  $L^m = L$ .

**Proof.** Let  $LB_{\rho_i} \in \mathcal{LB}_{\rho}$ . Then,  

$$L^m = C^m - E^m = \sum_{i \in N} \left( c_i - \sum_{k=1}^m Ls_i(LB_{\rho_i}^k) \right) - \left( E - \sum_{i \in N} \sum_{k=1}^m Ls_i(LB_{\rho_i}^k) \right) = C - E = L. \blacksquare$$

Second, for each  $P_i \in \{P_1, P_2, P_3\}$ , the order of the agents' claims remains the same along the *Recursive Lorenz P-Safety Process*.

**Remark 2** For each  $LB_{\rho_i} \in \mathcal{LB}_{\rho}$  with  $P_i \in \{P_1, P_2, P_3\}$ , and each  $i \in N$ ,

$$\text{if } c_i^m \leq c_j^m \text{ then } c_i^{m+1} \leq c_j^{m+1}.$$

**Proof.** Let  $LB_{\rho_i} \in \mathcal{LB}_{\rho}$  with  $P_i \in \{P_1, P_2, P_3\}$ , and let denote  $(CEA, CEL) = (\varphi^{D(LP_1)}, \varphi^{U(LP_1)})$ ,  $(Pin, DPin) = (\varphi^{D(LP_2)}, \varphi^{U(LP_2)})$  and  $(CE, DCE) = (\varphi^{D(LP_3)}, \varphi^{U(LP_3)})$ .

Since all of the above mentioned rules satisfy *Order Preservation*, then for each  $P_l \in \{P_1, P_2, P_3\}$ :

(a) If  $Ls_i(LB_{P_l}^m) = \varphi_i^{D(LP_l)}(E^m, c^m)$  and  $Ls_j(LB_{P_l}^m) = \varphi_j^{D(LP_l)}(E^m, c^m)$ , then, by *Order Preservation*,  $c_i^m - Ls_i(LB_{P_l}^m) \leq c_j^m - Ls_j(LB_{P_l}^m) \Rightarrow c_i^{m+1} \leq c_j^{m+1}$ .

(b) If  $Ls_i(LB_{P_l}^m) = \varphi_i^{D(LP_l)}(E^m, c^m)$  and  $Ls_j(LB_{P_l}^m) = \varphi_j^{U(LP_l)}(E^m, c^m)$ , by both *Order Preservation* and the definition of  $Ls_j(LB_{P_l}^m)$ ,  $c_i^m - \varphi_i^{D(LP_l)}(E^m, c^m) \leq c_j^m - \varphi_j^{U(LP_l)}(E^m, c^m) \leq c_j^m - \varphi_j^{D(LP_l)}(E^m, c^m) \leq c_j^m - \varphi_j^{U(LP_l)}(E^m, c^m) \Rightarrow c_i^{m+1} \leq c_j^{m+1}$ .

(c) If  $Ls_i(LB_{P_l}^m) = \varphi_i^{U(LP_l)}(E^m, c^m)$  and  $Ls_j(LB_{P_l}^m) = \varphi_j^{U(LP_l)}(E^m, c^m)$ , then, by *Order Preservation*,  $c_i^m - Ls_i(LB_{P_l}^m) \leq c_j^m - Ls_j(LB_{P_l}^m) \Rightarrow c_i^{m+1} \leq c_j^{m+1}$ . ■

Last remark says that, for each  $P_l \in \{P_1, P_2, P_3\}$ , the sum of the amounts that agents receive by the *Recursive Lorenz P-Safety Process* is the entire endowment.

**Remark 3** For each  $LB_{P_l} \in \mathcal{LB}_P$  with  $P_l \in \{P_1, P_2, P_3\}$ , and each  $i \in N$ ,

$$\sum_{i \in N} \left( \sum_{m=1}^{\infty} [RLS^m(LB_{P_l})]_i \right) = E.$$

**Proof.** Let  $LB_{P_l} \in \mathcal{LB}_P$  with  $P_l \in \{P_1, P_2, P_3\}$  and  $c^*$  and  $E^*$ , denote the limits of the sequences of the claims vectors and the endowment generated by the *Recursive Lorenz P-Safety Process*. These limits exist since claims and endowment do not increase from step to step, and all of these variables are bounded below by zero. We then show by contradiction that  $E^* = 0$ . Let us suppose that  $E^* > 0$ . Then, since for each  $P_l \in \{P_1, P_2, P_3\}$ ,  $Ls_j(LB_{P_l}^*) > 0$ , where  $j$  denotes the highest claimant, there is  $m \in \mathbb{N}$  such that  $E^m - E^* \leq Ls_j(LB_{P_l}^*)$ . Since  $LB_{P_l}^m$  is a well-defined *Lorenz-Bifocal Bankruptcy Problem*,  $Ls_j(LB_{P_l}^m) > 0$ , and given that all agents receive non-negative amounts, the endowment at the  $(m+1)$ -th step decreases by at least the agent  $j$ 's *Lorenz P-Safety*.

Therefore,  $E^{m+1} \leq E^m - Ls_j(LB_{P_l}^m) < E^m - Ls_j(LB_{P_l}^*) \leq E^*$ , which contradicts the definition of  $E^*$ . ■

## A.2 Proof of Theorem 1.

*Theorem 1: For each  $LB_{P_1} \in LB_P$ ,  $LB_{P_1} = ((E, c), CEA, CEL)$ , the Recursive Lorenz P-Safety rule is the Constrained Equal Losses rule,  $\varphi^{RL}(LB_{P_1}) = CEL(E, c)$ .*

The proof is based on four lemmas, but before presenting them, we note the following three facts. We assume, without loss of generality, that  $(E, c) \in \mathcal{B}_0$ .

**Fact 1** *For each  $(E, c) \in \mathcal{B}_0$  and each  $i \in N$ ,  $CEL_i(E, c) = \max\{0, c_i - \mu\}$ , where  $\mu$  is such that  $\sum_{i \in N} \max\{0, c_i - \mu\} = E$ .*

*Therefore,  $\mu$  can be understood as the losses incurred by the agents who receive positive amounts by applying the Constrained Equal Losses rule.*

A straightforward way to compute this rule, which will be useful later on, is as follows.

**Fact 2** *For each  $(E, c) \in \mathcal{B}_0$  and each  $i \in N$ , the loss imposed on agent  $i$  by CEL is*

$$\gamma_i = \min\{c_i, \alpha_i\},$$

where

$$\alpha_i = \left( L - \sum_{j < i} \gamma_j \right) / (n - i + 1).$$

Therefore, for each  $i \in N$ ,

$$CEL_i(E, c) = c_i - \gamma_i.$$

**Fact 3** *By Fact 1 and Remark 1 we get:*

- (a) *For each  $(E, c) \in \mathcal{B}_0$ , and each  $i \in N$ , if  $\gamma_i = c_i$ , then for each  $j < i$ ,  $\gamma_j = c_j$ .*
- (b) *For each  $(E, c) \in \mathcal{B}_0$ , and each  $i \in N$ , if  $\gamma_i = \alpha_i$ , then  $\alpha_i = \mu$  and for each  $j > i$ ,  $\alpha_j = \alpha_i$ . Therefore  $\gamma_i = \mu$  (where  $\mu$  is defined in the previous fact).*
- (c) *At each step  $m \in \mathbb{N}$  and for each  $i \in N$ ,  $\alpha_i^m$  only depends on the initial problem,  $(E, c)$ , and on agent  $j$ 's claim, for each  $j < i$ , at step  $m$ .*

Next, we provide the four lemmas on which Theorem 1 is based.

**Lemma 1** For each  $LB_{P_1} \in \mathcal{LB}_P$ , such that  $(E, c) \in \mathcal{B}_0$ , and each  $m \in \mathbb{N}$ ,  $\mu^{m+1} = \mu^m$ .

**Proof.** Let agent  $i$  be the first agent who receives positive amounts at step  $m \in \mathbb{N}$ , i.e., (i)  $Ls_i(LB_{P_1}^m) > 0$  and (ii) for each  $j < i$ ,  $Ls_j(LB_{P_1}^m) = 0$ . By (i) and Fact 3,  $c_i^m > \mu^m = \alpha_i^m$ . Given (ii) and the definition of the *Recursive Lorenz P-Safety Process* (Definition 7) at the  $m$ -th step,  $c_j^{m+1} = c_j^m$ . By Fact 3-(c),  $\alpha_i^{m+1} = \alpha_i^m = \mu^m < c_i^m$ . Furthermore,

$$\begin{aligned} c_i^{m+1} &= c_i^m - \min \{CEL_i(E^m, c^m), CEA_i(E^m, c^m)\} \geq \\ &\geq c_i^m - CEL_i(E^m, c^m) = c_i^m - (c_i^m - \mu^m) = \\ &= \mu^m = \alpha_i^{m+1}. \end{aligned}$$

Therefore, by Remark 2 and Fact 3-(b),  $\gamma_i^{m+1} = \alpha_i^{m+1} = \mu^{m+1}$ . ■

From now on,  $\mu$  will denote  $\mu^m$ , for each  $m \in \mathbb{N}$ .

The second lemma states that if at some step  $m \in \mathbb{N}$  the agent  $i$ 's *Lorenz P-Safety* for  $P_1$  is  $CEL_i(E^m, c^m)$ , then in each subsequent step, her *Lorenz P-Safety* for  $P_1$  is zero.

**Lemma 2** For each  $LB_{P_1} \in \mathcal{LB}_P$ , such that  $(E, c) \in \mathcal{B}_0$ , if there is  $m \in \mathbb{N}$  if  $Ls_i(LB_{P_1}^m) = CEL_i(E^m, c^m)$  then, for each  $h \in \mathbb{N}$ ,  $h > 0$ ,

$$Ls_i(LB_{P_1}^{m+h}) = 0.$$

**Proof.** Let  $LB_{P_1} \in \mathcal{LB}_P$ , such that  $(E, c) \in \mathcal{B}_0$ , and  $m \in \mathbb{N}$ , be such that

$$Ls_i(LB_{P_1}^m) = CEL_i(E^m, c^m) = c_i^m - \min \{c_i^m, \mu\}.$$

Then,

$$c_i^{m+1} = c_i^m - CEL_i(E^m, c^m) = c_i^m - (c_i^m - \min \{c_i^m, \mu\}) = \min \{c_i^m, \mu\}.$$

Therefore,

$$\begin{aligned}
 CEL_i(E^{m+1}, c^{m+1}) &= c_i^{m+1} - \min \{c_i^{m+1}, \mu\} = \\
 &= \min \{c_i^m, \mu\} - \min \{\min \{c_i^m, \mu\}, \mu\} = \\
 &= \min \{c_i^m, \mu\} - \min \{c_i^m, \mu\} = 0.
 \end{aligned}$$

And,  $Ls_i(LB_{\rho_i}^{m+1}) = CEL_i(E^{m+1}, c^{m+1}) = 0$ .

Thus, the agent  $i$ 's *Lorenz P-Safety* for  $P_1$  is, from this step on, zero because for each  $h \in \mathbb{N}$ ,  $h > 0$ ,  $CEL_i(E^{m+h}, c^{m+h}) = 0 = Ls_i(LB_{\rho_i}^{m+h})$ . ■

The next lemma establishes that, if agent  $i$ 's *Lorenz P-Safety* for  $P_1$  is, at each step, the amount provided by the *Constrained Equal Awards* rule, then the total amount received by this agent is at most her award as calculated by the *Constrained Equal Losses* rule applied to the initial problem.

**Lemma 3** For each  $LB_{\rho_i} \in \mathcal{LB}_{\rho}$ , such that  $(E, c) \in \mathcal{B}_0$ , and each  $i \in N$ , if, for each  $m \in \mathbb{N}$ ,

$$\begin{aligned}
 Ls_i(LB_{\rho_i}^m) &= CEA_i(E^m, c^m), \text{ then} \\
 \varphi_i^{RL}(LB_{\rho_i}) &= \sum_{k=1}^{\infty} Ls_i(LB_{\rho_i}^k) \leq CEL_i(E, c).
 \end{aligned}$$

**Proof.** Let  $LB_{\rho_i} \in \mathcal{LB}_{\rho}$ , such that  $(E, c) \in \mathcal{B}_0$ , and  $i \in N$ . If for each  $m \in \mathbb{N}$ ,  $Ls_i(LB_{\rho_i}^m) = CEA_i(E^m, c^m)$ , then by the definition of the *Lorenz P-Safety* (Definition 5), and given that  $CEL_i(E^m, c^m) \geq CEA_i(E^m, c^m)$ ,

$$Ls_i(LB_{\rho_i}^m) \leq CEL_i(E^m, c^m) = c_i^m - \mu = c_i - \sum_{k=1}^{m-1} Ls_i(LB_{\rho_i}^k) - \mu,$$

so that

$$Ls_i(LB_{\rho_i}^m) + \sum_{k=1}^{m-1} Ls_i(LB_{\rho_i}^k) \leq c_i - \mu,$$

that is,

$$\sum_{k=1}^m Ls_i(LB_{\rho_i}^k) \leq CEL_i(E, c).$$

Therefore,

$$\lim_{m \rightarrow \infty} \sum_{k=1}^m Ls_i(LB_{\rho_1}^k) \leq c_i - \mu = CEL_i(E, c). \quad \blacksquare$$

The last lemma says that if the agent's *Lorenz P-Safety* for  $P_1$  corresponds at some step  $m^* \in \mathbb{N}$  with the amount recommended by the *Constrained Equal Losses* rule for the problem  $(E^{m^*}, c^{m^*})$  and at the previous step is the amount provided by the *Constrained Equal Awards* rule for each problem  $(E^{m^*-1}, c^{m^*-1})$ , then the total amount received by this agent at step  $m^*$  is that given by the *Constrained Equal Losses* rule applied to the initial problem.

**Lemma 4** For each  $LB_{\rho_1} \in \mathcal{LB}_{\rho}$ , such that  $(E, c) \in \mathcal{B}_0$ , and each  $i \in N$ , if there is  $m^* \in \mathbb{N}$ ,  $m^* > 1$ , such that  $Ls_i(LB_{\rho_1}^{m^*}) = CEL_i(E^{m^*}, c^{m^*})$  and  $Ls_i(LB_{\rho_1}^{m^*-1}) = CEA_i(E^{m^*-1}, c^{m^*-1})$ , then,

$$\sum_{k=1}^{m^*} Ls_i(LB_{\rho_1}^k) = CEL_i(E, c).$$

**Proof.** Let  $LB_{\rho_1} \in \mathcal{LB}_{\rho}$ , such that  $(E, c) \in \mathcal{B}_0$ . We have

$$\begin{aligned} Ls_i(LB_{\rho_1}^{m^*}) &= CEL_i(E^{m^*}, c^{m^*}) \text{ and} \\ Ls_i(LB_{\rho_1}^{m^*-1}) &= CEA_i(E^{m^*-1}, c^{m^*-1}). \end{aligned}$$

Since  $CEA_i(E^{m^*-1}, c^{m^*-1}) > 0$ , then  $CEL_i(E^{m^*-1}, c^{m^*-1}) > 0$ . Therefore  $c_i^{m^*-1} > \mu$  and by Lemma 1,  $c_i^{m^*} \geq \mu$ .

Then, at step  $m^*$ , agent  $i$  received

$$\begin{aligned}
\sum_{k=1}^{m^*} Ls_i(LB_{\rho_1}^k) &= \sum_{k=1}^{m^*} Ls_i(LB_{\rho_1}^k) + CEL_i(E^{m^*}, c^{m^*}) = \\
&= \sum_{k=1}^{m^*} Ls_i(LB_{\rho_1}^k) + [c_i^{m^*} - \min\{c_i^{m^*}, \mu\}] = \\
&= \sum_{k=1}^{m^*} Ls_i(LB_{\rho_1}^k) + \left[ \left( c_i - \sum_{k=1}^{m^*} Ls_i(LB_{\rho_1}^k) \right) - \min\{c_i^{m^*}, \mu\} \right] = \\
&= c_i - \min\{c_i^{m^*}, \mu\} = c_i - \mu.
\end{aligned}$$

Therefore,

$$\sum_{k=1}^{m^*} Ls_i(LB_{\rho_1}^k) = CEL_i(E, c). \quad \blacksquare$$

### Proof of Theorem 1.

Let  $LB_{\rho_1} \in \mathcal{LB}_{\rho}$ , such that  $(E, c) \in \mathcal{B}_0$ . There are two cases.

**Case a:** All agents claim the same amount. Then, by definition of *Lorenz P-Safety* for  $P_1$ , each agent receives the same amount and the entire endowment is distributed at the first step. Therefore,  $\varphi^{RL}(LB_{\rho_1}) = CEL(E, c)$ .

**Case b:** There are at least two agents whose claims differ. By construction, in this case the agent with the smallest claim, say  $i$ , receives as *Lorenz P-Safety* for  $P_1$ ,  $Ls_i(LB_{\rho_1}) = CEL_i(E, c)$ . Furthermore, for all  $m \in \mathbb{N}$  the agent with the highest claim, say  $j$ , receives as *Lorenz P-Safety* for  $P_1$ ,  $Ls_j(LB_{\rho_1}) = CEA_j(E^m, c^m)$ . Now, by Lemmas 2 and 4 for each agent  $r \in N$  who at some step  $m \in \mathbb{N}$  receives  $CEL_r(E^m, c^m)$  as *Lorenz P-Safety* for  $P_1$ , we have  $\varphi_r^{RL}(LB_{\rho_1}) = CEL_r(E, c)$ . For each of the remaining agents, say  $l \neq r$ , by Lemma 3,  $\varphi_l^{RL}(LB_{\rho_1}) \leq CEL_l(E, c)$ . Then, since  $\varphi^{RL}(LB_{\rho_1})$  exhausts the endowment, by Remark 3,  $\varphi^{RL}(LB_{\rho_1}) = CEL(E, c)$ .  $\blacksquare$

### A.3 Proof of Theorem 2.

*Theorem 2:* For each  $LB_{P_2} \in LB_P$ ,  $LB_{P_2} = ((E, c), Pin, DPin)$ , the Recursive Lorenz P-Safety rule is the Dual of Piniles' rule,  $\varphi^{RL}(LB_{P_2}) = DPin(E, c)$ .

The proof is based on five lemmas, but before presenting them, we note four facts. In their proofs we assume, without loss of generality,  $(E, c) \in \mathcal{B}_0$ .

**Fact 4** The Dual of Piniles' rule can be written as follows, given  $(E, c) \in \mathcal{B}$ ,  $i \in N$ ,

$$DPin_i(E, c) = \begin{cases} \frac{c_i}{2} - \min \left\{ \frac{c_i}{2}, \lambda \right\} & \text{if } E \leq C/2 \\ \frac{c_i}{2} + \left( \frac{c_i}{2} - \min \left\{ \frac{c_i}{2}, \lambda \right\} \right) & \text{if } E \geq C/2 \end{cases},$$

where  $\lambda$  is such that  $\sum_{i \in N} DPin_i(E, c) = E$ .

**Fact 5** Given  $E^1 > C/2$  and  $E^2 = E^1 - C/2$ , the corresponding  $\lambda$ , computed for obtaining  $DPin$  rule, for  $(E^1, c)$  and  $(E^2, c)$ , denoted  $\lambda^1$  and  $\lambda^2$  respectively, are equal.

Indeed

$$\lambda^1 : \sum_{i \in N} \min \{c_i/2, \lambda^1\} = C - E^1$$

and

$$\lambda^2 : \sum_{i \in N} \min \{c_i/2, \lambda^2\} = C/2 - E^2 = C - E^1.$$

A straightforward way to compute the  $DPin$  rule, which will be useful later on, is as follows.

**Fact 6** For each  $(E, c) \in \mathcal{B}_0$ , and each  $i \in N$ , the loss imposed on agent  $i$  by  $DPin$  is

$$\gamma_i = \begin{cases} \frac{c_i}{2} + \min \{c_i/2, \alpha_i\} & \text{if } E \leq C/2 \\ \min \{c_i/2, \alpha_i^*\} & \text{if } E \geq C/2 \end{cases},$$

where

$$\alpha_i = \left( C/2 - E - \sum_{j < i} \gamma_j \right) / (n - i + 1),$$

and

$$\alpha_i^* = \left( L - \sum_{j < i} \gamma_j \right) / (n - i + 1).$$

Therefore, for each  $i \in N$ ,

$$DPin_i(E, c) = c_i - \gamma_i.$$

**Fact 7** By Fact 4 and Remark 1, we get:

(a) For each  $(E, c) \in \mathcal{B}_0$  and each  $i \in N$ ,

Case (i)  $E \leq C/2$ , if  $\gamma_i = c_i$  then, for each  $j < i$ ,  $\gamma_j = c_j$ .

Case (ii)  $E \geq C/2$ , if  $\gamma_i = c_i/2$  then, for each  $j < i$ ,  $\gamma_j = c_j/2$ .

(b) For each  $(E, c) \in \mathcal{B}_0$  and each  $i \in N$ , and given  $\lambda$  (defined in Fact 4),

Case (i)  $E \leq C/2$ , if  $\gamma_i = c_i/2 + \alpha_i$  then  $\alpha_i = \lambda$  and for each  $j > i$ ,  $\alpha_j = \alpha_i$ .

Therefore,  $\gamma_i = c_i/2 + \lambda$ .

Case (ii)  $E \geq C/2$ , if  $\gamma_i = \alpha_i$  then  $\alpha_i = \lambda$  and for each  $j > i$ ,  $\alpha_j = \alpha_i$ . Therefore,  $\gamma_i = \lambda$ .

(c) At each step  $m \in \mathbb{N}$  and for each  $i \in N$ ,  $\alpha_i^m$  only depends on the initial problem,  $(E, c)$ , and on agent  $j$ 's claim, for each  $j < i$ , at step  $m$ .

Next lemma shows that the relationship among claims and  $\lambda$  is fixed.

**Lemma 5** For each  $LB_{\rho_2} \in \mathcal{LB}_{\rho}$ , such that  $(E, c) \in \mathcal{B}_0$ , and each  $m \in \mathbb{N}$ ,

$$(a) \text{ if } c_i^m/2 < \lambda^m, \text{ then } c_i^{m+1}/2 < \lambda^m$$

and

$$(b) \text{ if } c_i^m/2 > \lambda^m, \text{ then } c_i^{m+1}/2 \geq \lambda^m.$$

**Proof.** Part (a) is obvious since  $c_i^{m+1} < c_i^m$ . In order to prove (b), we have two cases.

**Case b.1:**  $E \leq C/2$ . Let agent  $i$  be the first agent who receives a positive amount at step  $m \in \mathbb{N}$ , i.e., (i)  $Ls_i(LB_{\rho_2}^m) > 0$  and (ii) for each  $j < i$ ,  $Ls_j(LB_{\rho_2}^m) = 0$ . Given (ii) and the definition of the *Recursive Lorenz P-Safety Process* (Definition 7) at the  $m$ -th step,  $c_j^{m+1} = c_j^m$ .

Furthermore, since

$$\begin{aligned} c_i^{m+1}/2 &= c_i^m/2 - \min \{DPin_i(E^m, c^m), Pin_i(E^m, c^m)\} \geq \\ &\geq c_i^m/2 - DPin_i(E^m, c^m) = c_i^m/2 - (c_i^m/2 - \min \{c_i^m/2, \lambda^m\}) = \\ &= \min \{c_i^m/2, \lambda^m\} = \lambda^m. \end{aligned}$$

Therefore,

$$c_i^{m+1}/2 \geq \lambda^m.$$

**Case b.2:**  $E \geq C/2$ . Since agents receive their half-claim and the remaining endowment is distributed as previously (Fact 4) the conclusion follows directly. ■

The second lemma says that, in each step,  $\lambda^m$  is always the same.

**Lemma 6** For each  $LB_{\rho_2} \in \mathcal{LB}_{\rho}$ , such that  $(E, c) \in \mathcal{B}_0$ , and each  $m \in \mathbb{N}$ ,  $\lambda^m = \lambda^{m+1}$ .

**Proof.** By Fact 6, for each  $i \in N$ ,  $\gamma_i$  represents the loss incurred by agent  $i$ , and this will not change if  $\alpha_i$  does not change. By construction,  $\alpha_i$  is the same whenever the relationship among claims and  $\lambda$  is fixed. By Remark 2, Fact 7-(b) and Lemma 5, we know that  $\lambda^{m+1} = \gamma_i^{m+1} = \alpha_i^{m+1} = \alpha_i^m = \gamma_i^m = \lambda^m$ . ■

From now on,  $\lambda$  will denote  $\lambda^m$ ,  $\forall m \in \mathbb{N}$ .

The third lemma states that if at some step  $m \in \mathbb{N}$  agent's  $i$  Lorenz  $P$ -Safety for  $P_2$  is  $DPin_i(E^m, c^m)$ , then in the following steps, her Lorenz  $P$ -Safety for  $P_2$  is zero.

**Lemma 7** For each  $LB_{P_2} \in \mathcal{LB}_P$ , such that  $(E, c) \in \mathcal{B}_0$ , if there is  $m \in \mathbb{N}$  such that  $Ls_i(LB_{P_2}^m) = DPin_i(E^m, c^m)$  then,

$$Ls_i(LB_{P_2}^{m+h}) = 0, \text{ for each } h \in \mathbb{N}, h > 0.$$

**Proof.** Let  $LB_{P_2} \in \mathcal{LB}_P$ , such that  $(E, c) \in \mathcal{B}_0$ , and  $m \in \mathbb{N}$  be such that

$$Ls_i(LB_{P_2}^m) = DPin_i(E^m, c^m).$$

We will show that  $Ls_i(LB_{P_2}^{m+1}) = DPin_i(E^{m+1}, c^{m+1}) = 0$ .

**Case a:**  $E^m \leq C^m/2$  and  $\lambda \geq c_i^m/2$ . Then

$$DPin_i(E^m, c^m) = 0$$

and

$$c_i^{m+1}/2 = c_i^m/2 - (DPin_i(E^m, c^m)) = c_i^m/2.$$

Therefore,

$$DPin_i(E^{m+1}, c^{m+1}) = c_i^{m+1}/2 - \min\{c_i^{m+1}/2, \lambda\} = 0.$$

**Case b:**  $E^m \leq C^m/2$  and  $\lambda < c_i^m/2$ . Then

$$DPin_i(E^m, c^m) = c_i^m/2 - \min\{c_i^m/2, \lambda\} = c_i^m/2 - \lambda.$$

Thus,

$$\begin{aligned} c_i^{m+1}/2 &= c_i^m/2 - (DPin_i(E^m, c^m)) = \\ &= c_i^m/2 - (c_i^m/2 - \lambda) = \lambda. \end{aligned}$$

Therefore,

$$\begin{aligned} DPin_i(E^{m+1}, c^{m+1}) &= c_i^{m+1}/2 - \min \{c_i^{m+1}/2, \lambda\} = \\ &= \lambda - \min \{\lambda, \lambda\} = 0. \end{aligned}$$

**Case c:**  $E^m \geq C^m/2$ . Since agents receive their half-claims and the remaining endowment is distributed as previously (see Fact 4) the conclusion follows straightforwardly.

$$\text{And, } Ls_i(LB_{\rho_2}^{m+1}) = DPin_i(E^{m+1}, c^{m+1}) = 0.$$

Thus, the agent  $i$ 's *Lorenz P-Safety* for  $P_2$  is, from this step on, zero because for each  $h \in \mathbb{N}$ ,  $h > 0$ ,  $DPin_i(E^{m+h}, c^{m+h}) = 0 = Ls_i(LB_{\rho_2}^{m+h})$ . ■

Next lemma states that, if agent  $i$ 's *Lorenz P-Safety* for  $P_2$  is, at every step, the amount provided by *Piniles'* rule, then the total amount received by this agent will be at most her award corresponding to the *Dual of Piniles'* rule applied to the initial problem.

**Lemma 8** For each  $LB_{\rho_2} \in \mathcal{LB}_{\rho_2}$ , such that  $(E, c) \in \mathcal{B}_0$ , and each  $i \in N$ , if  $Ls_i(LB_{\rho_2}^m) = Pin_i(E^m, c^m)$  for each  $m \in \mathbb{N}$ , then

$$\varphi_i^{RL}(LB_{\rho_2}) = \sum_{k=1}^{\infty} Ls_i(LB_{\rho_2}^k) \leq DPin_i(E, c).$$

*Proof.* Let  $LB_{\rho_2} \in \mathcal{LB}_{\rho}$ , such that  $(E, c) \in \mathcal{B}_0$ , and  $i \in N$ ,

**Case a:** For each  $m \in \mathbb{N}$ ,  $E^m \leq C^m/2$ .

If for each  $m \in \mathbb{N}$   $LS_i(LB_{\rho_2}^m) = Pin_i(E^m, c^m)$ , then by the definition of the *Lorenz P-Safety* (Definition 5),

$$LS_i(LB_{\rho_2}^m) \leq DPin_i(E^m, c^m) = c_i^m/2 - \lambda = c_i/2 - \sum_{k=1}^{m-1} LS_i(LB_{\rho_2}^k) - \lambda,$$

then

$$LS_i(LB_{\rho_2}^m) + \sum_{k=1}^{m-1} LS_i(LB_{\rho_2}^k) \leq c_i/2 - \lambda,$$

that is,

$$\sum_{k=1}^m LS_i(LB_{\rho_2}^k) \leq DPin_i(E, c).$$

Therefore,

$$\lim_{k \rightarrow \infty} \sum_{k=1}^m LS_i(LB_{\rho_2}^k) \leq c_i/2 - \lambda = DPin_i(E, c)$$

**Case b:**  $E^m \geq C^m/2$ .

This case is only possible when  $m = 1$ . Now, since each agent receives her half-claim and the remaining endowment is distributed as in case 1 (See Fact 4), the proof follows straightforwardly. ■

Last lemma states that if each agent  $i$ 's *Lorenz P-Safety* for  $P_2$  at some step  $m^* \in \mathbb{N}$  is the amount provided by the *Dual of Piniles'* rule for the problem  $(E^{m^*}, c^{m^*})$  and at the previous step is the amount provided by *Piniles'* rule for the problem  $(E^{m^*-1}, c^{m^*-1})$ , then the total amount received by this agent at step  $m^*$  is that given by the *Dual of Piniles'* rule applied to the initial problem.

**Lemma 9** For each  $LB_{\rho_2} \in \mathcal{LB}_{\rho}$ , such that  $(E, c) \in \mathcal{B}_0$ , and each  $i \in N$ , if there is  $m^* \in \mathbb{N}$ ,  $m^* > 1$ , such that  $LS_i(LB_{\rho_2}^{m^*}) = DPin_i(E^{m^*}, c^{m^*})$  and

$$LS_i(LB_{\rho_2}^{m^*-1}) = Pin_i(E^{m^*-1}, c^{m^*-1}),$$

then

$$\sum_{k=1}^{m^*} LS_i(LB_{\rho_2}^k) = DPin_i(E, c).$$

*Proof.* Let  $LB_{\rho_2} \in \mathcal{LB}_{\rho}$ , such that  $(E, c) \in \mathcal{B}_0$ .

**Case a:**  $E \leq C/2$ . Then,

$$\begin{aligned} Ls_i(LB_{\rho_2}^{m^*}) &= DPin_i(E^{m^*}, c^{m^*}), \\ Ls_i(LB_{\rho_2}^{m^*-1}) &= Pin_i(E^{m^*-1}, c^{m^*-1}), \end{aligned}$$

by construction,  $Pin_i(E^{m^*-1}, c^{m^*-1}) > 0$  and by the definition of the *Lorenz P-Safety* (Definition 5),

$$DPin_i(E^{m^*-1}, c^{m^*-1}) > 0.$$

Given Fact 4,  $c_i^{m^*-1}/2 > \lambda$  and by Lemma 6,  $c_i^{m^*}/2 \geq \lambda$ . Then, at step  $m^*$ ,  $E^{m^*} \leq C^{m^*}/2$  and agent  $i$  received

$$\begin{aligned} \sum_{k=1}^{m^*} Ls_i(LB_{\rho_2}^k) &= \sum_{k=1}^{m^*-1} Ls_i(LB_{\rho_2}^k) + DPin_i(E^{m^*}, c^{m^*}) = \\ &= \sum_{k=1}^{m^*-1} Ls_i(LB_{\rho_2}^k) + [c_i^{m^*}/2 - \min\{c_i^{m^*}/2, \lambda\}] = \\ &= \sum_{k=1}^{m^*-1} Ls_i(LB_{\rho_2}^k) + \left[ \left( c_i/2 - \sum_{k=1}^{m^*-1} Ls_i(LB_{\rho_2}^k) \right) - \lambda \right] = \\ &= c_i/2 - \lambda = DPin_i(E, c). \end{aligned}$$

**Case b:**  $E \geq C/2$ . By Fact 4, each agent receives at least half of her claim, so  $\sum_{k=1}^{m^*} Ls_i(LB_{\rho_2}^k) = c_i/2 + \sum_{k=1}^{m^*} Ls_i(\tilde{E}^k, c^k, P_2)$ , where  $\tilde{E} = E - C/2$ .

Then, we follow the same reasoning than in case a, taking into account  $\tilde{E}$ .

So that,  $\tilde{E} \leq C/2$ , and,  $\tilde{LB}_{\rho_2} = (\tilde{E}, c, Pin, DPin)$ . Then,

$$\begin{aligned} Ls_i(\tilde{LB}_{\rho_2}^{m^*}) &= DPin_i(\tilde{E}^{m^*}, c^{m^*}), \\ Ls_i(\tilde{LB}_{\rho_2}^{m^*-1}) &= Pin_i(\tilde{E}^{m^*-1}, c^{m^*-1}), \end{aligned}$$

then, since by construction,  $Pi n_i(\tilde{E}^{\sim m^* - 1}, c^{m^* - 1}) > 0$ , by the definition of the *Lorenz P-Safety* (Definition 5),

$$DPi n_i(\tilde{E}^{\sim m^* - 1}, c^{m^* - 1}) > 0.$$

Therefore, by Fact 4 and Lemma 6,  $c_i^{m^* - 1}/2 > \lambda$ , which implies  $c_i^{m^*}/2 \geq \lambda$ .

Furthermore,  $\tilde{E}^{\sim m^*} \leq C^{m^*}/2$ .

Then, at step  $m^*$ , agent  $i$  received

$$\begin{aligned} \sum_{k=1}^{m^*} Ls_i(\tilde{LB}_{P_2}^{\sim k}) &= \sum_{k=1}^{m^* - 1} Ls_i(\tilde{LB}_{P_2}^{\sim k}) + DPi n_i(\tilde{E}^{\sim m^*}, c^{m^*}) = \\ &= \sum_{k=1}^{m^* - 1} Ls_i(\tilde{LB}_{P_2}^{\sim k}) + [c_i^{m^*}/2 - \min\{c_i^{m^*}/2, \lambda\}] = \\ &= \sum_{k=1}^{m^* - 1} Ls_i(\tilde{LB}_{P_2}^{\sim k}) + \left[ \left( c_i/2 - \sum_{k=1}^{m^* - 1} Ls_i(\tilde{LB}_{P_2}^{\sim k}) \right) - \lambda \right] = \\ &= c_i/2 - \lambda = DPi n_i(E, c). \end{aligned}$$

Therefore,

$$\sum_{k=1}^{m^*} Ls_i(LB_{P_2}^k) = c_i/2 + \sum_{k=1}^{m^*} Ls_i(\tilde{LB}_{P_2}^{\sim k}) = c_i - \lambda = DPi n_i(E, c). \blacksquare$$

### Proof of Theorem 2.

Let  $LB_{P_2} \in \mathcal{LB}_P$ , such that  $(E, c) \in \mathcal{B}_0$ . There are two cases.

**Case a:** All agents claim the same amount. Then, by definition of the *Lorenz P-Safety* for  $P_2$ , each agent receives the same amount and the entire endowment is distributed at the first step. So that,  $\varphi^{RL}(LB_{P_2}) = DPi n(E, c)$ .

**Case b:** There are at least two agents whose claims differ. By construction, given Fact 4, the agent with the smallest claim, say  $i$ , receives as *Lorenz P-Safety* for  $P_2$ ,  $Ls_i(LB_{P_2}) = DPi n_i(E, c)$ . Furthermore, for each  $m \in \mathbb{N}$  the agent with the

highest claim, say  $j$ , receives as *Lorenz P-Safety* for  $P_2$ ,  $Ls_j(LB_{P_2}^m) = Pin_j(E^m, c^m)$ . Now, by Lemmas 7 and 9, for each agent  $\tau \in N$  who at some step  $m \in \mathbb{N}$  receives  $DPin_\tau(E^m, c^m)$  as *Lorenz P-Safety* for  $P_2$ , we have  $\varphi_\tau^{RL}(LB_{P_2}) = DPin_\tau(E, c)$ . For each of the remaining agents, say  $l \neq \tau$ , by Lemma 8,  $\varphi_l^{RL}(LB_{P_2}) \leq DPin_l(E, c)$ . Then, we know, by Remark 3, that  $\varphi^{RL}(LB_{P_2})$  exhausts the endowment, thus,  $\varphi^{RL}(LB_{P_2}) = DPin(E, c)$ . ■

#### A.4 Proof of Theorem 3.

*Theorem 3: For each bi-personal  $LB_{P_3} \in LB_P$ ,  $LB_{P_3} = ((E, c), CE, DCE)$ , the Recursive Lorenz P-Safety rule is the Dual Constrained Egalitarian rule,  $\varphi^{RL}(LB_{P_3}) = DCE(E, c)$ .*

We note two facts, in which we assume, without loss of generality, that  $(E, c) \in \mathcal{B}_0$ .

**Fact 8** *The Dual Constrained Egalitarian rule can be written as follows,*

given  $(E, c) \in \mathcal{B}_0$ ,  $i \in N$ ,

$$DCE_i(E, c) \equiv \begin{cases} c_i - \max\{c_i/2, \min\{c_i, \delta\}\} & \text{if } E \leq C/2 \\ c_i - \min\{c_i/2, \delta\} & \text{if } E \geq C/2 \end{cases},$$

where  $\delta$  is chosen such that  $\sum_{i \in N} DCE_i(E, c) = E$ .

By Fact 8, it is obvious that agent one receives nothing if  $\min\{c_2 - c_1, c_2/2\} \geq E$ .

The following fact gives us two conditions that will be used in the proof of Theorem 3.

**Fact 9** *Let  $LB_{P_3} \in \mathcal{LB}_P$ , such that  $(E, c) \in \mathcal{B}_0$ , a bi-personal problem. Then, given Fact 8 and the definition of the Dual Constrained Egalitarian rule (see Section 0.1 of Preliminaries), at any step  $m \in \mathbb{N}$ ,*

$Ls_1(LB_{P_3}^m) = DCE_i(E^m, c^m)$ . Therefore:

(i) Next inequality characterizes the fact that agent one is guaranteed nothing at each step  $m \in \mathbb{N}$

$$Ls_1(LB_{P_3}^m) = 0 \Leftrightarrow \min \{c_2^m - c_1^m, c_2^m/2\} \geq E^m. \quad (\text{C.A.1})$$

(ii) The previous characterization for step  $m \in \mathbb{N}$  implies the following conditions in terms of the problem at step  $m - 1$ ,

$$\begin{aligned} E^m \leq c_2^m - c_1^m &\Leftrightarrow E^{m-1} - Ls_1(E^{m-1}, c^{m-1}, P_3) - Ls_2(E^{m-1}, c^{m-1}, P_3) \leq \\ &\leq c_2^{m-1} - Ls_2(E^{m-1}, c^{m-1}, P_3) - (c_1^{m-1} - Ls_1(E^{m-1}, c^{m-1}, P_3)) \end{aligned}$$

so that, for  $m = 2$ ,

$$E^2 \leq c_2^2 - c_1^2 \Leftrightarrow E \leq c_2 - c_1 + 2Ls_1(LB_{P_3}). \quad (\text{C.A.2.})$$

Furthermore,  $E^2 \leq c_2^2/2 \Leftrightarrow E - Ls_1(LB_{P_3}) - Ls_2(LB_{P_3}) \leq c_2/2 - Ls_2(LB_{P_3})/2$

and then,

$$E^2 \leq c_2^2/2 \Leftrightarrow E \leq c_2/2 + Ls_2(LB_{P_3})/2 + Ls_1(LB_{P_3}). \quad (\text{C.A.3})$$

### Proof of Theorem 3

For each bi-personal problem  $LB_{P_3} \in \mathcal{LB}_P$ , such that  $(E, c) \in \mathcal{B}_0$ , by Fact 8, at each step  $m \in \mathbb{N}$ ,  $Ls_1(LB_{P_3}^m) = DCE_1(E^m, c^m)$ , and  $Ls_2(LB_{P_3}^m) = CE_2(E^m, c^m)$ .

Now, we show that agent 1's *Lorenz P-Safety* for  $P_3$  at each step  $m \geq 2$ , is zero, so agent one's *Recursive Lorenz P-Safety* rule for  $P_3$  is the *Dual Constrained Egalitarian* rule. Then, since  $\varphi^{RL}(LB_{P_3})$  exhausts the endowment, given Remark 3,  $\varphi^{RL}(LB_{P_3}) = DCE_2(E, c)$ .

If  $c_1 = c_2$ , by the definition of the *Recursive Lorenz P-Safety* rule for  $P_3$  (Definition 8), each agent  $i$  receives the same amount at the initial step. And if  $c_1 \neq c_2$ , with  $E = (c_1 + c_2)/2$  by the *Midpoint Property*, each agent  $i$  receives

her half-claim,  $c_i/2$ . In both cases, therefore, at the initial step the endowment is exhausted, and  $\varphi^{RL}(LB_{P_3}) = DCE(E, c)$ .

When  $c_1 \neq c_2$  there are three cases.

**Case 1:**  $Ls_1(LB_{P_3}) = 0$ . Note that this is only possible in Region I.a of Figure A.1 and in Regions I.b and II.b of Figure A.2.

Then, by Condition C.A.1  $E \leq \min\{c_2 - c_1, c_2/2\}$ . Now, in the following step  $E^2 = E - Ls_2(LB_{P_3})$ ,  $c_1^2 = c_1$  and  $c_2^2 = c_2 - Ls_2(LB_{P_3})$ . Therefore, Condition C.A.1 again states that  $Ls_1(LB_{P_3}^2) = 0$  if and only if

$$E - Ls_2(LB_{P_3}) \leq c_2 - c_1 - Ls_2(LB_{P_3}), \text{ which follows from } E \leq c_2 - c_1,$$

and

$$E - Ls_2(LB_{P_3}) \leq (c_2/2) - (Ls_2(LB_{P_3})/2), \text{ which follows from } E \leq c_2/2.$$

Applying the previous reasoning from step 2 to the next step, and so on, we obtain  $\varphi_1^{RL}(LB_{P_3}) = 0$ . Therefore, by Remark 3,  $\varphi^{RL}(LB_{P_3}) = (0, E) = DCE(E, c)$ .

In Cases 2 and 3, we will show that at  $m = 2$  agent 1's *Lorenz P-Safety* for  $P_3$  is zero. Case 1 can then be applied to the residual *Lorenz-Bifocal Bankruptcy Problem*, so from  $m = 2$  on,  $Ls_1(E^{m+h}, c^{m+h}, P_3) = 0$ , for each  $h \in \mathbb{N}$ , and  $\varphi_1^{RL}(LB_{P_3}) = Ls_1(LB_{P_3})$ .

**Case 2:**  $Ls_1(LB_{P_3}) > 0$ , and  $c_2/2 \geq c_2 - c_1$ .

In this case the agents' *Lorenz P-Safety* for  $P_3$  can be placed in Regions II.a, III.a, IV.a and V.a of Figure A.1.

**Region II.a:**  $c_2 - c_1 \leq E \leq c_1$ . Then,  $Ls_1(LB_{P_3}) = (E + c_1 - c_2)/2$  and  $Ls_2(LB_{P_3}) = E/2$ . Conditions C.A.2. and C.A.3 imply  $E \leq 2c_1$ , which is true, as in this region,  $E \leq c_1$ . Therefore,

$$\varphi^{RL}(LB_{P_3}) = ((E + c_1 - c_2)/2, (E - c_1 + c_2)/2) = DCE(E, c).$$

**Region III.a:**  $c_1 \leq E \leq (c_1 + c_2)/2$ . Then,  $Ls_1(LB_{P_3}) = E - c_2/2$  and  $Ls_2(LB_{P_3}) = E - c_1/2$ . Now, Conditions C.A.2. and C.A.3 imply  $E \geq c_1$ , which is obviously fulfilled in this region. Therefore,

$$\varphi^{RL}(LB_{P_3}) = (E - (c_2/2), c_2/2) = DCE(E, c).$$

**Region IV.a:**  $(c_1 + c_2)/2 \leq E \leq [(c_1 + c_2)/2] + [(c_2 - c_1)/2] = c_2$ . Then,  $Ls_1(LB_{P_3}) = c_1/2$  and  $Ls_2(LB_{P_3}) = c_2/2$ . Again, from Conditions C.A.2. and C.A.3 we need to show both  $E \leq c_2$ , which is the Estate-upper bound of this region, and  $E \leq (3c_2/4) + (c_1/2)$ , which, by the Estate-upper bound of this region, is true since  $c_2/2 \geq c_2 - c_1$  in Case 2, which implies  $c_1/2 \geq c_2/4$ . Therefore,

$$\varphi^{RL}(LB_{P_3}) = (c_1/2, E - c_1/2) = DCE(E, c).$$

**Region V.a:**  $c_2 \leq E \leq 2c_1$ . Then,  $Ls_1(LB_{P_3}) = (E + c_1 - c_2)/2$  and  $Ls_2(LB_{P_3}) = E/2$ . Now, Conditions C.A.2. and C.A.3 imply  $E \leq 2c_1$ , which is obviously fulfilled in this region. Therefore,

$$\varphi^{RL}(LB_{P_3}) = ((E + c_1 - c_2)/2, (E - c_1 + c_2)/2) = DCE(E, c).$$

**Region VI.a:**  $2c_1 \leq E$ . Then,  $Ls_1(LB_{P_3}) = (E + c_1 - c_2)/2$  and  $Ls_2(LB_{P_3}) = E - c_1$ . Here, Conditions C.A.2. and C.A.3 do not imply any restriction, so that,

$$\varphi^{RL}(LB_{P_3}) = ((E + c_1 - c_2)/2, (E - c_1 + c_2)/2) = DCE(E, c).$$

**Case 3:**  $Ls_1(LB_{P_3}) > 0$ , and  $c_2/2 \leq c_2 - c_1$ . In this case, the agents' *Lorenz P-Safety* for  $P_3$  can be placed in Regions II.b, III.b, IV.b and V.b of Figure A.2.

**Region III.b:**  $c_2/2 \leq E \leq (c_1 + c_2)/2$ . Then,  $Ls_1(LB_{P_3}) = E - c_2/2$  and  $Ls_2(LB_{P_3}) = E - c_1/2$ . Conditions C.A.2. and C.A.3 imply  $E \geq c_1$ , inequality fulfilled as in this region  $c_2/2 \leq c_2 - c_1$ , implying  $c_1 \leq c_2/2$ . Therefore,

$$\varphi^{RL}(LB_{P_3}) = (E - c_2/2, c_2/2) = DCE(E, c).$$

**Region IV.b:**  $(c_1 + c_2)/2 \leq E \leq c_1 + c_2/2$ . Then  $Ls_1(LB_{P_3}) = c_1/2$  and  $Ls_2(LB_{P_3}) = c_2/2$ . Now, Conditions C.A.2. and C.A.3 imply  $E \leq c_2$  and  $E \leq$

$\leq 3c_2/4 + c_1/2$ , and both inequalities are satisfied as in this region  $c_2/2 \leq c_2 - c_1$ , which implies  $c_1 \leq c_2/2$ . Therefore,

$$\varphi^{RL}(LB_{\mathcal{P}_3}) = (c_1/2, E - c_1/2) = DCE(E, c).$$

**Region V.b:**  $c_1 + c_2/2 \leq E \leq c_2$ .  $Ls_1(LB_{\mathcal{P}_3}) = c_1/2$  and  $Ls_2(LB_{\mathcal{P}_3}) = E - c_1$ . Conditions C.A.2. and C.A.3 imply  $E \leq c_2$ , which is the Estate-upper bound in this region. Therefore,

$$\varphi^{RL}(LB_{\mathcal{P}_3}) = (c_1/2, E - c_1/2) = DCE(E, c).$$

**Region VI.b:**  $c_2 \leq E$ . Then,  $Ls_1(LB_{\mathcal{P}_3}) = (E + c_1 - c_2)/2$  and  $Ls_2(LB_{\mathcal{P}_3}) = E - c_1$ . Here, Conditions C.A.2. and C.A.3 do not imply any restriction, so that,

$$\varphi^{RL}(LB_{\mathcal{P}_3}) = ((E + c_1 - c_2)/2, (E - c_1 + c_2)/2) = DCE(E, c).$$

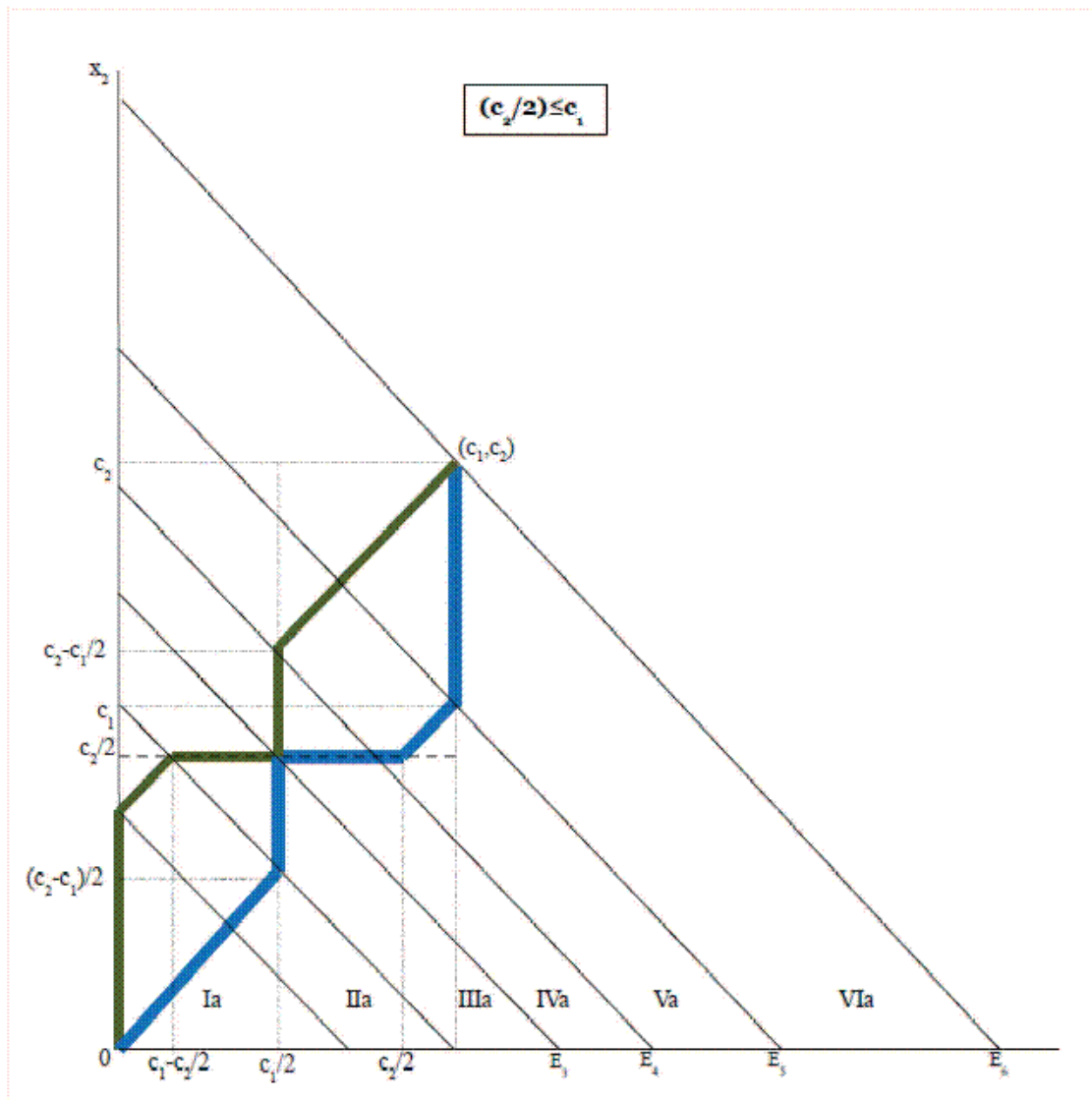


Figure A.1: Regions of the *Constrained Egalitarian* rule and its dual (case a).

Note that this figure the black solid lines represent all the possible sharing of six different levels of the estate ( $E_1$ ,  $E_2$ ,  $E_3$ ,  $E_4$ ,  $E_5$  and  $E_6$ ). The blue and the green solid lines show *CE* and *DCE*. The regions are, starting from the estate zero, those areas bounded by an estate and by the next one.

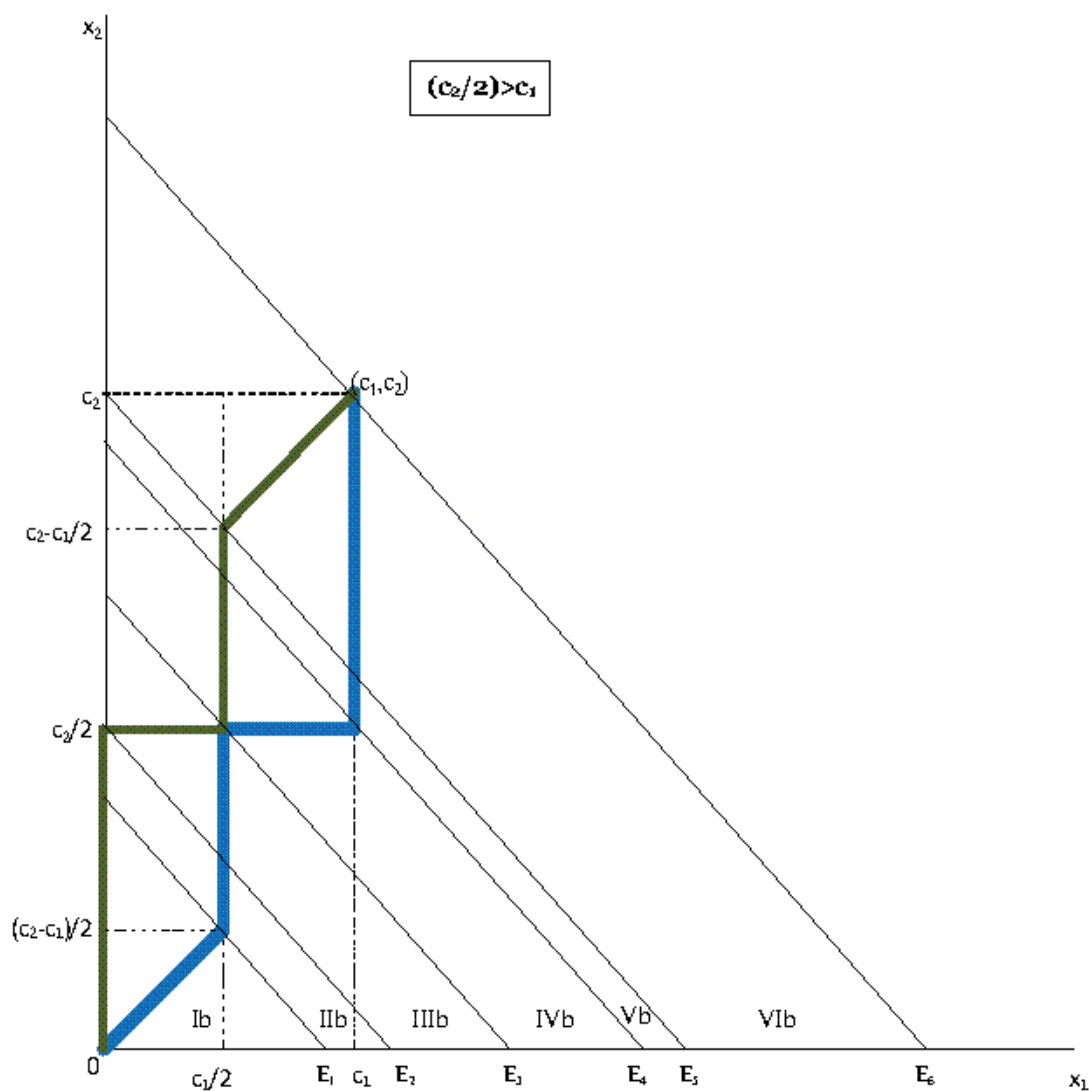


Figure A.2: Regions of the *Constrained Egalitarian* rule and its dual (case b).

Again, note that in this figure the black solid lines represent all the possible sharing of six different levels of the estate ( $E_1$ ,  $E_2$ ,  $E_3$ ,  $E_4$ ,  $E_5$  and  $E_6$ ). The dots and squares show *CE* and *DCE*. The regions are, starting from the estate zero, those areas bounded by an estate and by the next one. ■

## A.5 Proofs of Propositions 1 and 2.

*Proposition 1: There is a problem,  $(E, c) \in B$ , for which the Recursive Lorenz P-Safety rule for  $P_3$  does not coincide with the Dual Constrained Egalitarian rule,  $\varphi^{RL}(LB_{P_3}) \neq DCE(E, c)$ .*

*Proposition 2: For  $P_3$ , the Recursive Lorenz P-Safety rule does not satisfy Resource Monotonicity.*

First, we note the following fact, in which we assume, without loss of generality,  $(E, c) \in \mathcal{B}_0$ .

**Fact 10** *By Fact 8, the Dual Constrained Egalitarian rule can be written as follows, given  $(E, c) \in \mathcal{B}_0$ ,  $i \in N$ ,*

$$DCE_i(E, c) \equiv c_i - \gamma_i,$$

where  $\gamma_i$  is chosen such that  $\sum_{i \in N} DCE_i(E, c) = E$ .

Therefore,

**Case a:**  $E \leq C/2$ . We can compute  $\gamma_i$  as:

$$\gamma_i = \begin{cases} c_i & \forall i < l \\ \max\{c_i/2, \alpha_i\} & \forall i \geq l \end{cases},$$

where agent  $l$  is such that  $\sum_{j>l} \min\{c_j - c_l; c_j/2\} < E$ , and

either  $\sum_{j>l-1} \min\{c_j - c_{l-1}; c_j/2\} \geq E$ , either  $l = 1$ . Otherwise,  $l = n$ .

Then, for each  $i \geq l$ ,

$$\alpha_i = \frac{L - \sum_{j=1}^{l-1} c_j - \sum_{j>i} \gamma_j}{i - l + 1}.$$

Note also that we should compute  $\alpha$  from the highest claimant to the smallest one.

*Case b:*  $E \geq C/2$ . Then,  $\gamma_i$  denotes the losses incurred by agent  $i$  when the losses from the claim vector are equal for all agents subject to no-one obtaining less than her half-claim.

### Proof of Proposition 1.

Let us consider the problem  $LB_{P_3} \in LB_P$ , such that  $(E, c) = (21, (5; 19.5; 20)) \in \mathcal{B}$ .

Thus, given the definitions of the *Constrained Egalitarian* rule and its dual (Section 0.1 of Preliminaries), the definition of the *Lorenz P-Safety* (Definition 5) for  $P_3$ , and Fact 10, we obtain at step  $m = 1$ ,  $(E^1, c^1) = (21, (5; 19.5; 20))$ ,  $CE(E^1, c^1) = (2.5; 9.25; 9.25)$ , and  $DCE(E^1, c^1) = (5 - \gamma_1; 19.5 - \gamma_2; 20 - \gamma_3)$ .

Since  $\min\{20 - 5; 10\} + \min\{19.5 - 5; 9.75\} = 19.75 < E$ ,  $l = 1$ . Thus,

$$\alpha_3 = 23.5/3 = 7.83 \Rightarrow \gamma_3 = \max\{10; 7.83\} = 10.$$

$$\alpha_2 = (23.5 - 10)/2 = 6.75 \Rightarrow \gamma_2 = \max\{9.75; 6.75\} = 9.75.$$

$$\alpha_1 = 23.5 - 10 - 9.75 = 3.75 \Rightarrow \gamma_1 = \max\{2.5; 3.75\} = 3.75.$$

So,  $DCE(E^1, c^1) = (1.25; 9.75; 10)$ . Then,  $L_S(E^1, c^1, P_3) = (1.25; 9.25; 9.25)$ .

Step  $m = 2$ :

$(E^2, c^2) = (1.25, (3.75; 10.25; 10.75))$ ,  $CE(E^2, c^2) = (0.416; 0.416; 0.416)$ , and  $DCE(E^2, c^2) = (3.75 - \gamma_1^2; 10.25 - \gamma_2^2; 10.75 - \gamma_3^2)$ .

Since  $\min\{10.75 - 10.25; 5.375\} = 0.5 < E^2 = 1.25$ ,  $l = 2$ , and

for  $l = 1$ ,  $\min\{10.75 - 3.75; 5.375\} + \min\{10.25 - 3.75; 5.125\} = 10.5 > E^2$ .

Thus,

$$\alpha_3^2 = (23.5 - 3.75)/2 = 9.875 \Rightarrow \gamma_3^2 = \max\{5.375; 9.875\} = 9.875.$$

$$\alpha_2^2 = 23.5 - 3.75 - 9.875 = 9.875 \Rightarrow \gamma_2^2 = \max\{5.125; 9.875\} = 9.875.$$

$$\gamma_1^2 = 3.75.$$

So,  $DCE(E^2, c^2) = (0; 0.375; 0.875)$ . Then,  $Ls(LB_{P_3}^2) = (0; 0.375; 0.416)$ ,

and

$$\sum_{k=1}^2 Ls(LB_{P_3}^k) = (1.25; 9.625; 9.666).$$

Step  $m = 3$  :

$(E^3, c^3) = (0.459, (3.75; 9.875; 10.334))$ ,  $CE(E^3, c^3) = (0.153; 0.153; 0.153)$ , and  $DCE(E^3, c^3) = (3.75 - \gamma_1^3; 9.875 - \gamma_2^3; 10.334 - \gamma_3^3)$ .

$l = 2$  since  $\min\{10.334 - 9.875; 5.167\} = 0.459 = E^3$ , and

for  $l = 1$ ,  $\min\{10.334 - 3.75; 5.167\} + \min\{9.875 - 3.75; 4.937\} = 10.104 > E^3$ .

Thus,

$$\alpha_3^3 = 23.5 - 3.75 - 9.875 = 9.875 \Rightarrow \gamma_3^3 = \max\{5.167; 9.875\} = 9.875.$$

$$\gamma_2^3 = 9.875.$$

$$\gamma_1^3 = 3.75.$$

So,  $DCE(E^3, c^3) = (0; 0; 0.459)$ . Then,  $Ls(E^3, c^3, P_3) = (0; 0; 0.153)$ , and  $\sum_{k=1}^3 Ls(LB_{P_3}^k) = (1.25; 9.625; 9.819)$ .

Step  $m \geq 3$  :  $DCE(E^m, c^m) = (0; 0; E^m)$ , since  $E^m \leq c_3^m - c_2^m$ .

Then,  $\varphi^{RL}(LB_{P_3}) = (1.25; 9.625; 10.125) \neq DCE(E, c) = (1.25; 9.75; 10)$ . ■

### Proof of Proposition 2.

Let us consider the problem  $LB_{P_3}$  and  $LB'_{P_3} \in LB_{P_3}$ , such that  $(E, c) = (21, (5; 19.5; 20))$  and  $(E', c) = (22.25; (5; 19.5; 20)) \in \mathcal{B}$ , respectively.

In this case, given the definitions of the *Constrained Egalitarian* rule and its dual (see Section 0.1 of Preliminaries), and by the definition of the *Lorenz P-Safety* (Definition 5 for  $P_3$ ), and since the *Midpoint Property* is fulfilled,

$$\varphi^{RL}(LB'_{P_3}) = (2.5; 9.75; 10).$$

Moreover, in the previous example, we have seen that

$$\varphi^{RL}(LB_{\mathcal{F}_3}) = (1.25; 9.625; 10.125).$$

Obviously, these two distributions contradict *Resource Monotonicity* as the highest claimant receives less when the endowment increases. ■

## APPENDIX B

## PROOFS OF CHAPTER 3.

## B.1 Proof of Theorem 4.

*Theorem 4:* For each  $LB_{P_i} \in \mathcal{LB}_P$ , such that  $P_i$  is *Self-Dual*, then for each  $i \in N$ ,

$$\varphi_i^{LDR}(LB_{P_i}) = \frac{Lc_i(LB_{P_i}) + Ls_i(LB_{P_i})}{2}.$$

The proof of this result is based on two lemmas and a remark.

The first lemma shows that, in any step  $m \in \mathbb{N}$ ,  $m > 1$ , the sum of the *Lorenz P-Safety* and the *Lorenz P-Ceiling* coincides with the sum of the claims.

**Lemma 10** For each  $LB_{P_i} \in \mathcal{LB}_P$ , such that  $P_i$  is *Self-Dual*, and  $m \in \mathbb{N}$ ,  $m > 1$ ,

$$\sum_{i \in N} [Lc_i(LB_{P_i}^m) + Ls_i(LB_{P_i}^m)] = C^m.$$

**Proof.** Let  $LB_{P_i} \in \mathcal{LB}_P$ , such that  $P_i$  is *Self-Dual*, and  $m \in \mathbb{N}$ ,  $m > 1$ .

Note that for each  $i \in N$ , and each *Lorenz-Bifocal Admissible rule*,  $\varphi$ ,

$$\min \{LGM_i^{P_i}(E, c), LLM_i^{P_i}(E, c)\} \leq \varphi_i \leq \max \{LGM_i^{P_i}(E, c), LLM_i^{P_i}(E, c)\}.$$

Then, by the definitions of the *Lorenz P-Safety* and the *Lorenz P-Ceiling* (Definitions 5 and 9), these two rules define the *Lorenz P-Safety* and the *Lorenz P-Ceiling* of each agent for a set of properties  $P_i$ , i.e.,

$$Ls_i(LB_{P_i}) = \min \{LGM_i^{P_i}(E, c), LLM_i^{P_i}(E, c)\}, \text{ and}$$

$$Lc_i(LB_{P_i}) = \max \{LGM_i^{P_i}(E, c), LLM_i^{P_i}(E, c)\}.$$

Moreover, by their dual relation, for each agent we are adding the two *Lorenz-Focal rules*. So next expression comes straightforwardly.

$$\sum_{i \in N} \left[ \frac{Lc_i(LB_{P_i}^m) + Ls_i(LB_{P_i}^m)}{2} \right] = E^m.$$

Finally, we know that

$$\begin{aligned} E^m &= E^{m-1} - \sum_{i \in N} Ls_i(LB_{P_i}^{m-1}) = \\ &= \sum_{i \in N} \left[ \frac{Lc_i(LB_{P_i}^{m-1}) + Ls_i(LB_{P_i}^{m-1})}{2} \right] - \sum_{i \in N} Ls_i(LB_{P_i}^{m-1}) = \\ &= \sum_{i \in N} \left[ \frac{Lc_i(LB_{P_i}^{m-1}) - Ls_i(LB_{P_i}^{m-1})}{2} \right] = C^m/2, \end{aligned}$$

by the definition of the *Lorenz Double Boundedness Recursive Process* (Definition 10). ■

The following remark is a direct consequence of Lemma 10 and it says that for each *Lorenz-Bifocal Bankruptcy Problem*, and at any step  $m \in \mathbb{N}, m > 1$ , the half of the claims sum at every step of the *Lorenz Double Boundedness Recursive Process* (Definition 10) coincides with both the endowment and the total loss at every step of the process.

**Remark 4** For each  $(LB_{P_i}) \in \mathcal{LB}_\rho$ , such that  $P_i$  is *Self-Dual*, and  $m \in \mathbb{N}, m > 1$ ,  $E^m = L^m = C^m/2$ .

**Proof.** Let  $LB_{P_i} \in \mathcal{LB}_\rho$ , such that  $P_i$  is *Self-Dual*, and  $m > 1 \in \mathbb{N}$ . We know that,  $L^m = C^m - E^m$ . By Lemma 10,  $E^m = C^m/2$ . Therefore,  $L^m = C^m - C^m/2 = C^m/2$ . ■

The second lemma says that, each agent's claim at each step different of the initial one coincides with sum of both the lower and upper bound on awards.

**Lemma 11** For each  $LB_{\rho_i} \in \mathcal{LB}_{\rho}$ , such that  $P_i$  is *Self-Dual*, and  $m > 1 \in \mathbb{N}$ ,

$$c_i^m = Lc_i(LB_{\rho_i}^m) + Ls_i(LB_{\rho_i}^m).$$

**Proof.** Let  $LB_{\rho_i} \in \mathcal{LB}_{\rho}$ , such that  $P_i$  is *Self-Dual*, for each  $i \in N$ , and each  $m > 1 \in \mathbb{N}$ , by Remark 4 we know that for  $m > 1 \in \mathbb{N}$ ,  $L^m = E^m$ , so,  $Ls_i(LB_{\rho_i}^m) = Ls_i((LB_{\rho_i}^m)^d)$ . By duality  $Lc_i(LB_{\rho_i}^m) = c_i^m - Ls_i((LB_{\rho_i}^m)^d) = c_i^m - Ls_i(LB_{\rho_i}^m)$ , then,  $c_i^m = Lc_i(LB_{\rho_i}^m) + Ls_i(LB_{\rho_i}^m)$ . ■

#### Proof of Theorem 4.

Let  $LB_{\rho_i} \in \mathcal{LB}_{\rho}$ , such that  $P_i$  is *Self-Dual*, for each  $i \in N$ , and each  $m \in \mathbb{N}$ ,

$$\varphi_i^{LDR} LB_{\rho_i} = Ls_i(LB_{\rho_i}) + \sum_{m=2}^{\infty} Ls_i(LB_{\rho_i}^m).$$

By the definition of the *Lorenz Double Boundedness Recursive Process* (Definition 10),

$$\begin{aligned} \sum_{m=2}^{\infty} c_i^m &= \sum_{m=2}^{\infty} [Lc_i(LB_{\rho_i}^{m-1}) - Ls_i(LB_{\rho_i}^{m-1})] = \\ &= Lc_i(LB_{\rho_i}) + \sum_{m=2}^{\infty} Lc_i(LB_{\rho_i}^m) - Ls_i(LB_{\rho_i}) - \sum_{m=2}^{\infty} Ls_i(LB_{\rho_i}^m). \end{aligned}$$

By Lemma 11,

$$\sum_{m=2}^{\infty} c_i^m = \sum_{m=2}^{\infty} [Lc_i(LB_{\rho_i}^m) + Ls_i(LB_{\rho_i}^m)].$$

So,

$$\begin{aligned} Lc_i(LB_{\rho_i}) + \sum_{m=2}^{\infty} Lc_i(LB_{\rho_i}^m) - Ls_i(LB_{\rho_i}) - \sum_{m=2}^{\infty} Ls_i(LB_{\rho_i}^m) &= \\ &= \sum_{m=2}^{\infty} [Lc_i(LB_{\rho_i}^m) + Ls_i(LB_{\rho_i}^m)]. \end{aligned}$$

Thus,

$$\sum_{m=2}^{\infty} Ls_i(LB_{\rho_i}^m) = (Lc_i(LB_{\rho_i}) - Ls_i(LB_{\rho_i})) / 2.$$

Therefore,

$$\begin{aligned} \varphi_i^{LDR}(LB_{\rho_i}) &= Ls_i(LB_{\rho_i}) + \frac{Lc_i(LB_{\rho_i}) - Ls_i(LB_{\rho_i})}{2} \\ &= \frac{Ls_i(LB_{\rho_i}) + Lc_i(LB_{\rho_i})}{2}. \blacksquare \end{aligned}$$

## APPENDIX C

## PROOFS OF CHAPTER 4.

## C.1 General remarks

Next we present three remarks, in which we consider, without loss of generality,  $(E, c) \in \mathcal{B}_0$ . These remarks, which comes straightforwardly from the definition of the *Lorenz P-Safety* (Definition 5) for the sets of equity principles  $P_1$ ,  $P_2$  and  $P_3$ , and the definitions of *CEA*, *CEL*, *Pin*, *DPin*, *CE* and *DCE* rules (see Section 0.1 of Preliminaries), shows the smallest and highest agents' *Lorenz P-Safety*, respectively.

**Remark 5** Given  $LB_{\rho_1} \in \mathcal{LB}_P$ ,  $LB_{\rho_1} = ((E, c), CEA, CEL)$ , and for each  $m \in \mathbb{N}$ ,  $Ls_1(LB_{\rho_1}^m) = CEL_1(E^m, c^m)$  and  $Ls_n(LB_{\rho_1}^m) = CEA_n(E^m, c^m)$ .

**Remark 6** Given  $LB_{\rho_2} \in \mathcal{LB}_P$ ,  $LB_{\rho_2} = ((E, c), Pin, DPin)$ , and for each  $m \in \mathbb{N}$ ,  $Ls_1(LB_{\rho_2}^m) = DPin_1(E^m, c^m)$  and  $Ls_n(LB_{\rho_2}^m) = Pin_n(E^m, c^m)$ .

**Remark 7** Given  $LB_{\rho_3} \in \mathcal{LB}_P$ ,  $LB_{\rho_3} = ((E, c), CE, DCE)$ , and for each  $m \in \mathbb{N}$ ,  $Ls_1(LB_{\rho_3}^m) = DCE_1(E^m, c^m)$  and  $Ls_n(LB_{\rho_3}^m) = CE_n(E^m, c^m)$ .

## C.2 Proof of Proposition 5.

*Proposition 5:* For each  $LB_{\rho_2} \in \mathcal{LB}_P$ ,  $LB_{\rho_2} = ((E, c), Pin, DPin)$ , and each  $i \in \mathbb{N}$ , if  $\varphi^i(E, c) \in \Phi(LB_{\rho_2})$ , and for some  $j \in \mathbb{N}$ ,  $\varphi^j(E, c) = DPin(E, c)$ , then  $u[\psi, LB_{\rho_2}] = DPin(E, c)$ .

[Step 1] If all agents agree on  $\varphi(LB_{\rho_2}) = DPin(E, c)$ , then  $u[\psi, LB_{\rho_2}] = DPin(E, c)$ . Otherwise, go to next step.

[Step 2] Let  $Ls_i(LB_{\rho_2}) = \min_{j \in N} \varphi_i^j(LB_{\rho_2})$ ,  $c^2 = c - Ls(LB_{\rho_2})$ , and  $E^2 = E - \sum_{i \in N} Ls_i(LB_{\rho_2})$ . By Lemma 7, for each agent  $i$  such that  $Ls_i(LB_{\rho_2}) = DPin_i(E, c)$ ,  $Ls_i(LB_{\rho_2}^2) = 0$ . Since  $\varphi^j = DPin$ , if all agents agree on  $\varphi(LB_{\rho_2}^2)$ , by Lemma 9,  $DPin(E, c) \equiv u[\psi, LB_{\rho_2}] = Ls(LB_{\rho_2}) + \varphi(LB_{\rho_2}^2)$ . Otherwise, go to next step.

[Step  $m + 1$ ] Let  $Ls_i(LB_{\rho_2}^m) = \min_{j \in N} \varphi_i^j(LB_{\rho_2}^m)$ ,  $E^{m+1} = E^m - \sum_{i \in N} Ls_i(LB_{\rho_2}^m)$ , and  $c^{m+1} = c^m - Ls(LB_{\rho_2}^m)$ . By Lemma 7, for each agent  $i$  such that  $Ls_i(LB_{\rho_2}^m) = DPin_i(E^m, c^m)$ ,  $Ls_i(LB_{\rho_2}^{m+1}) = 0$ . Since  $\varphi^j = DPin$ , if all agents agree on  $\varphi(LB_{\rho_2}^{m+1})$ , by Lemma 9,  $DPin(E, c) \equiv u[\psi, LB_{\rho_2}] = \sum_{k=1}^m Ls(LB_{\rho_2}^k) + \varphi(LB_{\rho_2}^{m+1})$ . Otherwise, go to next step.

[Limit case] Compute  $\sum_{k=1}^{\infty} Ls_i(LB_{\rho_2}^k)$ . Note that, by Lemmas 7 and 9 and the definition of the  $DPin$  rule (Section 0.1 of Preliminaries), for each agent  $i \in N$  such that  $c_i \neq c_n$ ,  $\sum_{k=1}^{\infty} Ls_i(LB_{\rho_2}^k) = DPin_i(E, c)$ . Moreover, for the rest of agents,  $l$ , by Lemma 8,  $\sum_{k=1}^m Ls_l(LB_{\rho_2}^k) \leq DPin_l(E, c)$ . Furthermore, by Remark 6 and the definition of the  $DPin$  rule,

$$Ls_l(LB_{\rho_2}) \geq E/n \geq DPin_l(E, c)/n; \quad Ls_l(LB_{\rho_2}^2) \geq (DPin_l(E, c) - Ls_l(LB_{\rho_2}))/n;$$

thus,  $\sum_{k=1}^m Ls_l(LB_{\rho_2}^k) \geq \frac{DPin_l(E, c)}{n} \sum_{k=0}^m \left(\frac{n-1}{n}\right)^k$ , i.e.,

$$\sum_{k=1}^{\infty} Ls_l(LB_{\rho_2}^k) \geq DPin_l(E, c).$$

Therefore,  $\sum_{k=1}^{\infty} Ls_l(LB_{\rho_2}^k) = DPin_l(E, c)$ . ■

### C.3 Proof of Proposition 6.

*Proposition 6:* In the game  $\Gamma_{LB_{\rho_2}}^u$ , the  $DPin$  rule is a weakly dominant strategy for the agent with the highest claim.

By Remark 2, for each  $m \in \mathbb{N}$ ,  $c_1^m \leq c_2^m \leq \dots \leq c_n^m$ .

Moreover, note that for each  $LB_{\rho_2} \in \mathcal{LB}_{\rho}$  and each  $\varphi \in \Phi(LB_{\rho_2})$ ,

$$DPin_n(E, c) \geq \varphi_n(E, c).$$

Finally, by Lemmas 7, 8 and 9,  $\sum_{k=1}^{\infty} Ls_n(LB_{\rho_2}^k) \leq DPin_n(E, c)$ .

Therefore,  $DPin_n(E, c) \geq u_n[\psi, LB_{\rho_2}]$ , i.e., the *DPin* rule is a weakly dominant strategy for the agent with the highest claim. ■

#### C.4 Proof of Theorem 6.

*Theorem 6:* In any Nash equilibrium induced by the game  $\Gamma_{LB_{\rho_2}}^u$ , each agent receives the amount given by the *DPin* rule.

Let us consider  $LB_{\rho_2} \in \mathcal{LB}_{\rho}$ . Then, each agent's outcome in any Nash equilibrium of  $\Gamma_{LB_{\rho_2}}^u$  satisfies  $DPin_i(E, c) \leq u_i[\psi, LB_{\rho_2}]$ , for each  $i \in N$ . Otherwise if for some  $i \in N$ ,  $DPin_i(E, c) > u_i[\psi, LB_{\rho_2}]$  then, by Proposition 5, agent  $i$  could deviate to choose *DPin*, which gives her more awards, contradicting the Nash equilibrium. Finally, if for each  $i \in N$ ,  $DPin_i(E, c) \leq u_i[\psi, LB_{\rho_2}]$ , then,  $u[\psi, LB_{\rho_2}] = DPin(E, c)$ , since  $\sum_{i \in N} u_i[\psi, LB_{\rho_2}] \leq E$ . ■

#### C.5 Proof of Proposition 7.

*Proposition 7:* For each  $LB_{\rho_3} \in \mathcal{LB}_{\rho}$ ,  $LB_{\rho_3} = ((E, c), CE, DCE)$ , with  $|N| = 2$ , and each  $i \in \{1, 2\}$ , if  $\varphi^i(E, c) \in \Phi(LB_{\rho_3})$ , and for some  $j \in \{1, 2\}$ ,  $\varphi^j(E, c) = DCE(E, c)$ , then  $u[\psi, LB_{\rho_3}] = DCE(E, c)$ .

[Step 1] If the two agents agree on  $\varphi(LB_{\rho_3}) = DCE(E, c)$ , then  $u[\psi, LB_{\rho_3}] = DCE(E, c)$ . Otherwise, go to next step.

[Step 2] Let  $Ls_i(LB_{\rho_3}) = \min_{j \in N} \varphi_i^j(LB_{\rho_3})$ ,  $c^2 = c - Ls(LB_{\rho_3})$ , and  $E^2 = E - \sum_{i \in N} Ls_i(LB_{\rho_3})$ . In this case, by Remark 7,  $Ls_1(LB_{\rho_3}) = DCE_1(E, c)$ , and by Fact 9,  $Ls_1(LB_{\rho_3}^2) = 0$ . Since  $\varphi^j = DCE$ , if all agents agree on  $\varphi(LB_{\rho_3}^2)$ , then, as we can see in the proof of Theorem 3 (Appendix A.4),  $u[\psi, LB_{\rho_3}] = Ls(LB_{\rho_3}) + \varphi(LB_{\rho_3}^2) = DCE(E, c)$ . Otherwise, go to next step.

[Step  $m + 1$ ] Let  $Ls_i(LB_{\rho_3}^m) = \min_{j \in N} \varphi_i^j(LB_{\rho_3}^m)$ ,  $E^{m+1} = E^m - \sum_{i \in N} Ls_i(LB_{\rho_3}^m)$ , and  $c^{m+1} = c^m - Ls(LB_{\rho_3}^m)$ . By Fact 9,  $Ls_1(LB_{\rho_3}^m) = 0$ . Since  $\varphi^j = DCE$ , if all agents agree on  $\varphi(LB_{\rho_3}^m)$ , then, as we can see in the proof of Theorem 3 (Appendix A.4),  $u[\psi, LB_{\rho_3}] = \sum_{k=1}^m Ls(LB_{\rho_3}^k) + \varphi(LB_{\rho_3}^{m+1}) = DCE(E, c)$ . Otherwise, go to next step.

[Limit case] Compute  $\sum_{k=1}^{\infty} Ls_i(LB_{\rho_3}^k)$ . Note that, by Fact 9 and the definition of the  $DCE$  rule (Section 0.1 of Preliminaries), and, as we can see in the proof of Theorem 3 (Appendix A.4), for agent 1,  $\sum_{k=1}^{\infty} Ls_1(LB_{\rho_3}^k) = DPi m_1(E, c)$ .

Moreover, by Remark 7,  $\sum_{k=1}^m Ls_2(LB_{\rho_3}^k) \leq DCE_2(E, c)$ . Furthermore, by Fact 7 and the definition of the  $DCE$  rule,

$$\begin{aligned} Ls_2(LB_{\rho_3}) &\geq E/2 \geq DCE_2(E, c)/2; \\ Ls_2(LB_{\rho_3}^2) &\geq \frac{DCE_2(E, c) - Ls_2(LB_{\rho_3})}{2}, \end{aligned}$$

thus,

$$\begin{aligned} Ls_2(LB_{\rho_3}) + Ls_2(LB_{\rho_3}^2) &\geq Ls_2(LB_{\rho_3}) + \frac{DCE_2(E, c) - Ls_2(LB_{\rho_3})}{2} \\ &\geq \left(1 - \frac{1}{2}\right) Ls_2(LB_{\rho_3}) + \frac{DCE_2(E, c)}{2} \\ &\geq \left(1 - \frac{1}{2}\right) \frac{DCE_2(E, c)}{2} + \frac{DCE_2(E, c)}{2} \\ &\geq \left(1 + \frac{1}{2}\right) \frac{DCE_2(E, c)}{2} \end{aligned}$$

So that, in general,

$$\sum_{k=1}^m L_{S_2}(LB_{\rho_3}^k) \geq \frac{DCE_2(E, c)}{2} \sum_{k=0}^{m-1} \left(\frac{1}{2}\right)^k.$$

By considering the limit when  $m \rightarrow \infty$  we obtain

$$\sum_{k=1}^{\infty} L_{S_2}(LB_{\rho_3}^k) \geq DCE_2(E, c).$$

Therefore,  $\sum_{k=1}^{\infty} L_{S_2}(LB_{\rho_3}^k) = DCE_2(E, c)$ . ■

### C.6 Proof of Theorem 7.

*Theorem 7: For each  $LB_{\rho_3} \in LB_{\rho}$ ,  $LB_{\rho_3} = ((E, c), CE, DCE)$ , with  $|N| = 2$ , in any Nash equilibrium induced by the game  $\Gamma_{LB_{\rho_3}}^u$ , each agent receives the amount given by the DCE rule.*

Let us consider  $LB_{\rho_3} \in \mathcal{LB}_{\rho}$ . Then, each agent's outcome, in any Nash equilibrium of  $\Gamma_{LB_{\rho_3}}^u$  satisfies  $DCE_i(E, c) \leq u_i[\psi, LB_{\rho_3}]$  for each  $i \in N$ , with  $|N| = 2$ . Otherwise if for some  $i \in \{1, 2\}$ ,  $DCE_i(E, c) > u_i[\psi, LB_{\rho_3}]$  then, by Proposition 7, agent  $i$  could deviate to choose  $DCE$ , which gives her more awards, contradicting the Nash equilibrium. Finally, if for each  $i \in \{1, 2\}$ ,  $DCE_i(E, c) \leq u_i[\psi, LB_{\rho_3}]$ , then,  $u[\psi, LB_{\rho_3}] = DCE(E, c)$ , since  $\sum_{i \in N} u_i[\psi, LB_{\rho_3}] \leq E$ . ■

### C.7 Proof of Theorems 8 and 9.

*Theorem 8: There is a problem,  $(E, c) \in B$ , for which in a Nash equilibrium induced by the game  $\Gamma_{LB_{\rho_3}}^u$ , not all agents receive the amount given by the DCE rule.*

*Theorem 9: The Nash equilibrium induced by the game  $\Gamma_{LB_{\rho_3}}^u$  does not fulfill Resource Monotonicity.*

**Proof of Theorem 8.**

Let us consider the following problem  $LB_{\rho_3} \in \mathcal{LB}_\rho$ , such that,  $(E, c) \in \mathcal{B} = (21, (5; 19.5; 20))$ , and for each step  $m \in \mathbb{N}$ ,

$$\psi(LB_{\rho_3}^m) = (CE, \varphi^2, DCE).$$

Thus, given the definitions of the  $CE$  rule and its dual and Fact 10, we get at step  $m = 1$ ,  $(E^1, c^1) = (21, (5; 19.5; 20))$ ,  $CE(E^1, c^1) = (2.5; 9.25; 9.25)$ , and  $DCE(E^1, c^1) = (1.25; 9.75; 10)$ .

[Step 1] Since there is no agreement, go to next step.

[Step 2]  $Ls(LB_{\rho_3}) = (1.25; 9.25; 9.25)$ , and  $E^2 = 1.25$ . So,

$(E^2, c^2) = (1.25, (3.75; 10.25; 10.75))$ ,  $CE(E^2, c^2) = (0.416; 0.416; 0.416)$ , and  $DCE(E^2, c^2) = (0; 0.375; 0.875)$ , and since there is no agreement, go to next step.

[Step 3]  $Ls(LB_{\rho_3}^2) = (0; 0.375; 0.416)$ , and  $E^3 = 0.459$ . So,

$(E^3, c^3) = (0.459, (3.75; 9.875; 10.334))$ ,  $CE(E^3, c^3) = (0.153; 0.153; 0.153)$ , and  $DCE(E^3, c^3) = (0; 0; 0.459)$ , and since there is no agreement, go to next step.

[Step 4]  $Ls(LB_{\rho_3}^3) = (0; 0; 0.153)$ , and  $E^4 = 0.306$ . So,

$(E^4, c^4) = (0.306, (3.75; 9.875; 10.181))$ ,  $CE(E^4, c^4) = (0.102; 0.102; 0.102)$ , and  $DCE(E^4, c^4) = (0; 0; 0.306)$ , and since there is no agreement, go to next step.

[Limit case] Note that, since  $E^m \leq c_3^m - c_2^m$ , and  $c_3^m \geq E^m/3$ , for each step  $m \geq 3$ ,  $DCE(E^m, c^m) = (0; 0; E^m)$  and  $CE(E^m, c^m) = (E^m/3; E^m/3; E^m/3)$ . Thus,  $Ls(LB_{\rho_3}^m) = (0; 0; E^m/3)$ .

Therefore,  $u[\psi, LB_{\rho_3}] = \sum_{k=1}^{\infty} Ls(LB_{\rho_3}^k) = (1.25; 9.625; 10.125)$ .

At this point, we have to show that the  $CE$  and  $DCE$  are the weakly dominant strategies for agents 1 and 3 in this example, respectively.

In this regard, next we can observe that agent 3 cannot increase her payoff by changing her strategy.

Let us consider the same problem  $\bar{L}B_{\rho_3} \in \mathcal{LB}_{\rho}$ , such that,  $(E, c) \in \mathcal{B} = (21, (5; 19.5; 20))$ , and for each step  $m \in \mathbb{N}$ ,

$$\psi(\bar{L}B_{\rho_3}) = (CE, \varphi^2, \varphi^3),$$

where, since from step 2 on, the remaining estate in this case is different from the previous problem, we will denote by  $\bar{E}^m$  the estate for each step  $m > 1$ .

Thus, given the definitions of the  $CE$  rule and its dual (Section 0.1 of Preliminaries) and Fact 8, we get at step  $m = 1$ ,  $(E^1, c^1) = (21, (5; 19.5; 20))$ ,  $CE(E^1, c^1) = (2.5; 9.25; 9.25)$ , and  $DCE(E^1, c^1) = (1.25; 9.75; 10)$ .

[Step 1] It is obvious that there is no agreement, because if all the agents chooses the  $CE$  rule, then

$$u_3[\psi, \bar{L}B_{\rho_3}] = CE_3(E, c) = 9.25 \leq 10.125 = u_3[\psi, LB_{\rho_3}]$$

Therefore, go to next step.

[Step 2] By construction,

$$\begin{aligned} \varphi_1(\bar{L}B_{\rho_3}^1) &\geq DCE_1(E^1, c^1), \quad \varphi_1(\bar{L}B_{\rho_3}^1) \leq CE_1(E^1, c^1), \\ \varphi_2^3(\bar{L}B_{\rho_3}^1) &\geq CE_2(\bar{E}^1, c^1), \quad \varphi_2^2(\bar{L}B_{\rho_3}^1) \geq CE_2(\bar{E}^1, c^1), \\ \varphi_3(\bar{L}B_{\rho_3}^1) &\geq CE_3(\bar{E}^1, c^1), \quad \varphi_3(\bar{L}B_{\rho_3}^1) \leq DCE_2(E^1, c^1). \end{aligned}$$

Then,

$$Ls(\bar{L}B_{\rho_3}) = \left( \min \left\{ \varphi_1^2(\bar{L}B_{\rho_3}^1), \varphi_1^3(\bar{L}B_{\rho_3}^1) \right\}; 9.25; 9.25 \right).$$

Note that,

$$\min \left\{ \varphi_1^2(\bar{L}B_{\rho_3}^1), \varphi_1^3(\bar{L}B_{\rho_3}^1) \right\} = DCE_1(E^1, c^1) + \alpha, \text{ where } \alpha \in \mathbb{R}_+ \text{ such that } DCE_1(E^1, c^1) \leq DCE_1(E^1, c^1) + \alpha \leq CE_1(E^1, c^1),$$

$$\text{and } Ls_3(\bar{L}B_{\rho_3}) = Ls_3(LB_{\rho_3}).$$

Thus,

$$\bar{E}^2 = 1.25 - \alpha \leq E^2. \text{ So,}$$

$$\left(\bar{E}^2, c^2\right) = (1.25 - \alpha, (3.75 - \alpha; 10.25; 10.75)),$$

$$CE(\bar{E}^2, c^2) = (0.416 - \alpha/3; 0.416 - \alpha/3; 0.416 - \alpha/3), \text{ and}$$

$$DCE(\bar{E}^2, c^2) = (0; 0.375 - \alpha/2; 0.875 - \alpha/2), \text{ and since there is no agreement,}$$

go to next step.

[Step 3] By construction,

$$\varphi_1(\bar{LB}_{\rho_3}^2) \geq DCE_1(\bar{E}^2, c^2), \varphi_1(\bar{LB}_{\rho_3}^2) \leq CE_1(\bar{E}^2, c^2),$$

$$\varphi_2(\bar{LB}_{\rho_3}^2) \geq DCE_2(\bar{E}^2, c^2), \varphi_2(\bar{LB}_{\rho_3}^2) \leq CE_2(\bar{E}^2, c^2),$$

$$\varphi_3(\bar{LB}_{\rho_3}^2) \geq CE_3(\bar{E}^2, c^2), \varphi_3(\bar{LB}_{\rho_3}^2) \leq DCE_2(\bar{E}^2, c^2).$$

Then,

$$Ls(\bar{LB}_{\rho_3}^2) =$$

$$= \left( \min \left\{ \varphi_1^2(\bar{LB}_{\rho_3}^2), \varphi_1^3(\bar{LB}_{\rho_3}^2) \right\}; \min \left\{ \varphi_2^2(\bar{LB}_{\rho_3}^2), \varphi_2^3(\bar{LB}_{\rho_3}^2) \right\}; 0.416 - \alpha/3 \right).$$

Note that,

$$\min \left\{ \varphi_1^2(\bar{LB}_{\rho_3}^2), \varphi_1^3(\bar{LB}_{\rho_3}^2) \right\} = DCE_1(\bar{E}^2, c^2) + \alpha^2, \text{ where } \alpha^2 \in \mathbb{R}_+ \text{ such that}$$

$$DCE_1(\bar{E}^2, c^2) \leq DCE_1(\bar{E}^2, c^2) + \alpha^2 \leq CE_1(\bar{E}^2, c^2),$$

$$\min \left\{ \varphi_2^2(\bar{LB}_{\rho_3}^2), \varphi_2^3(\bar{LB}_{\rho_3}^2) \right\} = DCE_2(\bar{E}^2, c^2) + \varepsilon^2, \text{ where } \varepsilon^2 \in \mathbb{R}_+ \text{ such that}$$

$$DCE_2(\bar{E}^2, c^2) \leq DCE_2(\bar{E}^2, c^2) + \varepsilon^2 \leq CE_2(\bar{E}^2, c^2),$$

$$\text{and } Ls_3(\bar{LB}_{\rho_3}^2) \leq Ls_3(LB_{\rho_3}^2).$$

Thus,

$$\bar{E}^3 = (1.25 - \alpha) - \alpha^2 - (0.375 - \frac{\alpha}{2} - \varepsilon^2) - (0.416 - \frac{\alpha}{3}) = 0.459 - \frac{\alpha}{6} - \alpha^2 - \varepsilon^2 \leq E^3.$$

So,

$$\left(\bar{E}^3, c^3\right) = (0.459 - \frac{\alpha}{6} - \alpha^2 - \varepsilon^2, (3.75 - \alpha; 9.875 - \varepsilon^2; 10.334)),$$

$$CE(\bar{E}^3, c^3) = \left(\bar{E}^3/3; \bar{E}^3/3; \bar{E}^3/3\right), \text{ and}$$

$$DCE(\bar{E}^3, c^3) = (0; 0; \bar{E}^3), \text{ and since there is no agreement, go to next step.}$$

[Step 4] By construction,

$$\begin{aligned}\varphi_1(\bar{L}B_{\rho_3}^3) &\geq DCE_1(\bar{E}^3, c^3), \quad \varphi_1(\bar{L}B_{\rho_3}^3) \leq CE_1(\bar{E}^3, c^3), \\ \varphi_2(\bar{L}B_{\rho_3}^3) &\geq DCE_2(\bar{E}^3, c^3), \quad \varphi_2(\bar{L}B_{\rho_3}^3) \leq CE_2(\bar{E}^3, c^3), \\ \varphi_3(\bar{L}B_{\rho_3}^3) &\geq CE_3(\bar{E}^3, c^3), \quad \varphi_3(\bar{L}B_{\rho_3}^3) \leq DCE_2(\bar{E}^3, c^3).\end{aligned}$$

Then,

$$\begin{aligned}Ls(\bar{L}B_{\rho_3}^3) &= \\ &= \left( \min \left\{ \varphi_1^2(\bar{L}B_{\rho_3}^3), \varphi_1^3(\bar{L}B_{\rho_3}^3) \right\}; \min \left\{ \varphi_1^2(\bar{L}B_{\rho_3}^3), \varphi_1^3(\bar{L}B_{\rho_3}^3) \right\}; \bar{E}^3/3 \right).\end{aligned}$$

Note that,

$$\min \left\{ \varphi_1^2(\bar{L}B_{\rho_3}^3), \varphi_1^3(\bar{L}B_{\rho_3}^3) \right\} = DCE_1(\bar{E}^3, c^3) + \alpha^3, \text{ where } \alpha^3 \in \mathbb{R}_+ \text{ such that } DCE_1(\bar{E}^3, c^3) \leq DCE_1(\bar{E}^3, c^3) + \alpha^3 \leq CE_1(\bar{E}^3, c^3),$$

$$\min \left\{ \varphi_2^2(\bar{L}B_{\rho_3}^3), \varphi_2^3(\bar{L}B_{\rho_3}^3) \right\} = DCE_2(\bar{E}^3, c^3) + \varepsilon^3, \text{ where } \varepsilon^3 \in \mathbb{R}_+ \text{ such that } DCE_2(\bar{E}^3, c^3) \leq DCE_2(\bar{E}^3, c^3) + \varepsilon^3 \leq CE_2(\bar{E}^3, c^3),$$

$$\text{and } Ls_3(\bar{L}B_{\rho_3}^3) \leq Ls_3(LB_{\rho_3}^3).$$

Thus,

$$\bar{E}^4 = \bar{E}^3 - \alpha^3 - \varepsilon^3 - \bar{E}^3/3 \leq E^4.$$

Since there is no agreement, go to next step.

[Limit case] Note that, since  $\bar{E}^m \leq E^m$ , for each step  $m \geq 1$ .

Thus,  $u_3[\psi, \bar{L}B_{\rho_3}] = \sum_{k=1}^{\infty} Ls_3(\bar{L}B_{\rho_3}^k) \leq 10.125 = u_3[\psi, LB_{\rho_3}]$ , i.e., the *DCE* rule is a weakly dominant strategy for the agent 3.

Finally, by duality we can easily get that agent 1 cannot increase her payoff by changing her strategy, i.e., the *CE* rule is a weakly dominant strategy for the agent 1.

Therefore, since agent's 2 strategy does not influence in the procedure, in the *Nash equilibrium* induced by the game  $\Gamma_{LB_{\rho_3}}^u$  for this problem,  $u[\psi, LB_{\rho_3}] \neq DCE(E, c)$ . ■

**Proof of Theorem 9.**

Let us consider the problem  $LB_{\rho_3}$  and  $LB'_{\rho_3} \in LB_{\rho}$ , such that  $(E, c) = (21, (5; 19.5; 20))$  and  $(E', c) = (22.25; (5; 19.5; 20)) \in \mathcal{B}$ , respectively.

In this case, given the definitions of the *Constrained Egalitarian* rule and its dual (see Section 0.1 of Preliminaries), and since the *Midpoint Property* is fulfilled,

$$u[\psi, LB'_{\rho_3}] = (2.5; 9.75; 10),$$

Moreover, in the previous example, we have seen that

$$u[\psi, LB_{\rho_3}] = (1.25; 9.625; 10.125).$$

Obviously, these two distributions contradict *Resource Monotonicity* since the highest claimant receives less when the endowment increases. ■

**C.8 Proof of Theorem 11.**

*Theorem 11: In any Nash equilibrium induced by the game  $\Gamma_{LB_{\rho_i}}^{\text{du}}$ , such that  $P_i$  is Self-Dual, each agent receives the amount given by the average of the two Lorenz-Focal rules.*

Let  $LB_{\rho_i} \in \mathcal{LB}_{\rho}$ , such that  $P_i$  is *Self-Dual*.

By Theorem 4 we know that whenever each claimant's *Lorenz P-Safety* and *Lorenz P-Ceiling* corresponds with the amount recommending by one of the two *Lorenz-Focal rules*, then all the agents receive the amount given by the average of the two *Lorenz-Focal rules*.

Moreover, by the definition of the *Double Concessions* procedure (Definition 14), we can easily note that the *Lorenz-Gains Maximal* rule is a weakly dominant strategy for the smallest agent, and the *Lorenz-Losses Maximal* rule is a weakly dominant

strategy for the highest claimant. Thus,

$$Ls_i(LB_{\rho_i}^m) = \min_{\varphi \in \Phi(LB_{\rho_i})} \{LGM_i^{\rho_i}, LLM_i^{\rho_i}\}, \text{ and}$$

$$Lc_i(LB_{\rho_i}^m) = \max_{\varphi \in \Phi(LB_{\rho_i})} \{LGM_i^{\rho_i}, LLM_i^{\rho_i}\}.$$

Therefore, by Theorem 4,

$$\begin{aligned} du[\psi, LB_{\rho_i}] &= \lim_{m \rightarrow \infty} \sum_{k=1}^m Ls(LB_{\rho_i}^k) = Ls_i(LB_{\rho_i}) + \sum_{m=2}^{\infty} Ls_i(LB_{\rho_i}^m) \\ &= \frac{Ls_i(LB_{\rho_i}) + Lc_i(LB_{\rho_i})}{2}. \end{aligned}$$

■



## APPENDIX D

## PROOFS OF CHAPTER 5.

## D.1 Proof of Proposition 8.

*Proposition 8:* Given  $LB_{\rho_i} \in LB_{\rho}$ , if  $x \in C(V^{LB_{\rho_i}})$  then, for all  $i \in N$ ,

$$\min\{LGM_i^{\rho_i}(E, c), LLM_i^{\rho_i}(E, c)\} \leq x_i \leq \max\{LGM_i^{\rho_i}(E, c), LLM_i^{\rho_i}(E, c)\}.$$

Let  $LB_{\rho_i} \in LB_{\rho}$  and let  $x \in C(V^{LB_{\rho_i}})$ . Then, by definition of both the bankruptcy rule (Definition 2) and the *Core* distribution (Section 0.5 of Preliminaries),

$$\sum_{i \in N} LGM_i^{\rho_i}(E, c) = \sum_{i \in N} LLM_i^{\rho_i}(E, c) = \sum_{i \in N} x_i = V^{LB_{\rho_i}}(N) = E \quad (\text{D.1})$$

and

$$x_i \geq V^{LB_{\rho_i}}(\{i\}) = \min\{LGM_i^{\rho_i}(E, c), LLM_i^{\rho_i}(E, c)\} \text{ for all } i \in N.$$

Now, we only have to prove that  $x_i \leq \max\{LGM_i^{\rho_i}(E, c), LLM_i^{\rho_i}(E, c)\}$  for all  $i \in N$ .

Let us suppose that there exists  $i \in N$  such that

$$x_i > \max\{LGM_i^{\rho_i}(E, c), LLM_i^{\rho_i}(E, c)\} \text{ and,}$$

without loss of generality, let us assume that

$$LLM_i^{P_i}(E, c) \leq LGM_i^{P_i}(E, c).$$

Then

$$x_i > LGM_i^{P_i}(E, c). \quad (D.2)$$

Let  $S = \{j \in N \setminus \{i\}\}$ . Then, on the one hand, by Conditions D.1 and D.2,

$$\sum_{j \in S} x_j < \sum_{j \in S} LGM_j^{P_i}(E, c) \quad (D.3)$$

and on the other hand, since  $LLM_i^{P_i}(E, c) \leq LGM_i^{P_i}(E, c)$ , Condition D.1 implies

$$\sum_{j \in S} LLM_j^{P_i}(E, c) \geq \sum_{j \in S} LGM_j^{P_i}(E, c). \quad (D.4)$$

Therefore, by Conditions D.4 and D.3,

$$V^{LB_{P_i}}(S) = \min \left\{ \sum_{j \in S} LLM_j^{P_i}(E, c), \sum_{j \in S} LGM_j^{P_i}(E, c) \right\} = \sum_{j \in S} LGM_j^{P_i}(E, c) > \sum_{j \in S} x_j,$$

in contradiction with the fact that  $x \in \mathbb{C}(V^{LB_{P_i}})$ . Thus,

$$x_i \leq \max \{LGM_i^{P_i}(E, c), LLM_i^{P_i}(E, c)\} \text{ for all } i \in N. \blacksquare$$

## D.2 Proof of Proposition 9.

*Proposition 9:* Given  $LB_{P_i} \in LB_P$ , the associated Lorenz-Bifocal Bankruptcy Game,  $V^{LB_{P_i}}$ , is a PS-game such that for all  $i \in N$  and for all coalition  $T \subseteq N \setminus \{i\}$ ,

$$\Delta_i V^{LB_{P_i}}(T) + \Delta_i V^{LB_{P_i}}(N \setminus [T \cup \{i\}]) = LGM_i^{P_i}(E, c) + LLM_i^{P_i}(E, c).$$

Let  $LB_{\rho_i} \in \mathcal{LB}_{\rho}$ . By the definition of the *Lorenz-Bifocal Bankruptcy Game* (Definition 15),  $V^{LB_{\rho_i}}$ , is such that  $V^{LB_{\rho_i}}(\emptyset) = 0$  and, for each coalition  $S, \emptyset \neq S \subseteq N$ ,

$$V^{LB_{\rho_i}}(S) = \min \left\{ \sum_{i \in S} LGM_i^{\rho_i}(E, c), \sum_{i \in S} LLM_i^{\rho_i}(E, c) \right\}.$$

Let  $\alpha_i = LGM_i^{\rho_i}(E, c) - LLM_i^{\rho_i}(E, c)$  for all  $i \in N$ . Then, the worth of each coalition  $S \subseteq N$  can be expressed as follows:

$$V^{LB_{\rho_i}}(S) = \begin{cases} \sum_{j \in S} LLM_j^{\rho_i}(E, c) & \text{if } \sum_{j \in S} \alpha_j \geq 0 \\ \sum_{j \in S} LGM_j^{\rho_i}(E, c) & \text{if } \sum_{j \in S} \alpha_j \leq 0 \end{cases}. \quad (\text{D.5})$$

Next, we calculate the marginal contribution of any agent  $i \in N$  to any coalition  $T \subseteq N \setminus \{i\}$ , that is,  $\Delta_i V^{LB_{\rho_i}}(T) = V^{LB_{\rho_i}}(T \cup \{i\}) - V^{LB_{\rho_i}}(T)$ . The following four cases exhaust all the possibilities:

**Case 1** :  $\sum_{j \in T} \alpha_j \geq 0$  and  $\alpha_i + \sum_{j \in T} \alpha_j \geq 0$ .

By Condition D.5,  $V^{LB_{\rho_i}}(T) = \sum_{j \in T} LLM_j^{\rho_i}(E, c)$  and,

$V^{LB_{\rho_i}}(T \cup \{i\}) = \sum_{j \in T} LLM_j^{\rho_i}(E, c) + LLM_i^{\rho_i}(E, c)$ . Thus,

$$\Delta_i V^{LB_{\rho_i}}(T) = LLM_i^{\rho_i}(E, c). \quad (\text{D.6})$$

**Case 2** :  $\sum_{j \in T} \alpha_j \geq 0$  and  $\alpha_i + \sum_{j \in T} \alpha_j \leq 0$

By Condition D.5,  $V^{LB_{\rho_i}}(T) = \sum_{j \in T} LLM_j^{\rho_i}(E, c)$  and

$V^{LB_{\rho_i}}(T \cup \{i\}) = \sum_{j \in T} LGM_j^{\rho_i}(E, c) + LGM_i^{\rho_i}(E, c)$ . Thus,

$$\Delta_i V^{LB_{\rho_i}}(T) = LGM_i^{\rho_i}(E, c) + \sum_{j \in T} LGM_j^{\rho_i}(E, c) - \sum_{j \in T} LLM_j^{\rho_i}(E, c)$$

$$= LGM_i^{\rho_i}(E, c) + \sum_{j \in T} \alpha_j. \quad (D.7)$$

**Case 3** :  $\sum_{j \in T} \alpha_j \leq 0$  and  $\alpha_i + \sum_{j \in T} \alpha_j \leq 0$

By Condition D.5,  $V^{\mathcal{L}B\rho_i}(T) = \sum_{j \in T} LGM_j^{\rho_i}(E, c)$  and

$V^{\mathcal{L}B\rho_i}(T \cup \{i\}) = \sum_{j \in T} LGM_j^{\rho_i}(E, c) + LGM_i^{\rho_i}(E, c)$ . Thus,

$$\Delta_i V^{\mathcal{L}B\rho_i}(T) = LGM_i^{\rho_i}(E, c). \quad (D.8)$$

**Case 4** :  $\sum_{j \in T} \alpha_j \leq 0$  and  $\alpha_i + \sum_{j \in T} \alpha_j \geq 0$

By Condition D.5,  $V^{\mathcal{L}B\rho_i}(T) = \sum_{j \in T} LGM_j^{\rho_i}(E, c)$  and  $V^{\mathcal{L}B\rho_i}(T \cup \{i\}) = \sum_{j \in T} LLM_j^{\rho_i}(E, c) + LLM_i^{\rho_i}(E, c)$ . Thus,

$$\begin{aligned} \Delta_i V^{\mathcal{L}B\rho_i}(T) &= \\ &= LLM_i^{\rho_i}(E, c) + \sum_{j \in T} LLM_j^{\rho_i}(E, c) - \sum_{j \in T} LGM_j^{\rho_i}(E, c) = LLM_i^{\rho_i}(E, c) - \sum_{j \in T} \alpha_j. \end{aligned} \quad (D.9)$$

Next, we calculate the sum of the marginal contributions of any agent  $i \in N$  to any pair of disjoint coalitions  $T, T^*$  such that  $T \cup T^* = N \setminus \{i\}$ . With this aim, let us note that, given that both  $LGM^{\rho_i}$  and  $LLM^{\rho_i}$  are distribution rules,

$$\sum_{i \in N} \alpha_i = \sum_{i \in N} LGM_i^{\rho_i}(E, c) - \sum_{i \in N} LLM_i^{\rho_i}(E, c) = 0.$$

Therefore,

$$\sum_{k \in T^*} \alpha_k = - \left( \alpha_i + \sum_{j \in T} \alpha_j \right). \quad (D.10)$$

Here again we consider four cases for coalition  $T$ , which exhaust all the possibilities:

**Case 1** :  $\sum_{j \in T} \alpha_j \geq 0$  and  $\alpha_i + \sum_{j \in T} \alpha_j \geq 0$ . Then, by Condition D.10,  $\sum_{k \in T^*} \alpha_k \leq 0$  and

$\alpha_i + \sum_{k \in T^*} \alpha_k \leq 0$ . Now, applying Condition D.6 to coalition  $T$  and Condition

D.8 to coalition  $T^*$ , we have

$$\Delta_i V^{\mathcal{L}B\rho_i}(T) + \Delta_i V^{\mathcal{L}B\rho_i}(T^*) = LLM_i^{\rho_i}(E, c) + LGM_i^{\rho_i}(E, c).$$

**Case 2** :  $\sum_{j \in T} \alpha_j \geq 0$  and  $\alpha_i + \sum_{j \in T} \alpha_j \leq 0$ . Then, by Condition D.10,  $\sum_{k \in T^*} \alpha_k \geq 0$  and  $\alpha_i + \sum_{k \in T^*} \alpha_k \leq 0$ . Now, applying Condition D.7 to both coalitions  $T$  and  $T^*$ , we have

$$\begin{aligned} \Delta_i V^{LB_{\rho_i}}(T) + \Delta_i V^{LB_{\rho_i}}(T^*) &= LGM_i^{\rho_i}(E, c) + \sum_{j \in T} \alpha_j + LGM_i^{\rho_i}(E, c) + \sum_{k \in T^*} \alpha_k = \\ &= LGM_i^{\rho_i}(E, c) + LGM_i^{\rho_i}(E, c) - \alpha_i \\ &= LGM_i^{\rho_i}(E, c) + LGM_i^{\rho_i}(E, c) - LGM_i^{\rho_i}(E, c) + LLM_i^{\rho_i}(E, c) \\ &= LLM_i^{\rho_i}(E, c) + LGM_i^{\rho_i}(E, c). \end{aligned}$$

**Case 3** :  $\sum_{j \in T} \alpha_j \leq 0$  and  $\alpha_i + \sum_{j \in T} \alpha_j \leq 0$ . Then, by Condition D.10,  $\sum_{k \in T^*} \alpha_k \geq 0$  and  $\alpha_i + \sum_{k \in T^*} \alpha_k \geq 0$ . Now, applying Condition D.8 to coalition  $T$  and Condition D.6 to coalition  $T^*$ , we have

$$\Delta_i V^{LB_{\rho_i}}(T) + \Delta_i V^{LB_{\rho_i}}(T^*) = LGM_i^{\rho_i}(E, c) + LLM_i^{\rho_i}(E, c).$$

**Case 4** :  $\sum_{j \in T} \alpha_j \leq 0$  and  $\alpha_i + \sum_{j \in T} \alpha_j \geq 0$ . Then, by Condition D.10,  $\sum_{k \in T^*} \alpha_k \leq 0$  and  $\alpha_i + \sum_{k \in T^*} \alpha_k \geq 0$ . Now, applying Condition D.9 to both coalitions  $T$  and  $T^*$ , we have

$$\begin{aligned} \Delta_i V^{LB_{\rho_i}}(T) + \Delta_i V^{LB_{\rho_i}}(T^*) &= LLM_i^{\rho_i}(E, c) - \sum_{j \in T} \alpha_j + LLM_i^{\rho_i}(E, c) - \sum_{k \in T^*} \alpha_k = \\ &= LLM_i^{\rho_i}(E, c) + LLM_i^{\rho_i}(E, c) + \alpha_i \\ &= LLM_i^{\rho_i}(E, c) + LLM_i^{\rho_i}(E, c) + LGM_i^{\rho_i}(E, c) - LLM_i^{\rho_i}(E, c) \\ &= LLM_i^{\rho_i}(E, c) + LGM_i^{\rho_i}(E, c). \blacksquare \end{aligned}$$

### D.3 Proof of Theorem 12.

*Theorem 12: For each Lorenz-Bifocal Bankruptcy Game,  $V^{LB_{\rho_i}}$ , where  $LB_{\rho_i} \in \mathcal{LB}_{\rho}$ , the Shapley value and the Nucleolus coincide and they are obtained in the average of the two Lorenz-Focal rules, that is,*

$$Sh(V^{LB_{\rho_i}}) = Nu(V^{LB_{\rho_i}}) = 1/2 (LGM^{\rho_i}(E, c) + LLM^{\rho_i}(E, c)).$$

Taking into account Proposition 9 and applying, to  $V^{LB_{\rho_i}}$ , with  $LB_{\rho_i} \in \mathcal{LB}_{\rho}$ , the main result in Kar et al. [25], gathered below, we obtain that for all  $i \in N$ ,  $Sh_i(V^{LB_{\rho_i}}) = PNu_i(V^{LB_{\rho_i}}) = (LLM_i^{\rho_i}(E, c) + LGM_i^{\rho_i}(E, c))/2$ , where  $PNu$  denotes the Prenucleolus. Now, given that, by the definition of  $V^{LB_{\rho_i}}$ ,  $PNu(V^{LB_{\rho_i}})$  satisfies individual rationality, that is,  $PNu(V^{LB_{\rho_i}}) \geq V^{LB_{\rho_i}}(\{i\})$  for all  $i \in N$ , we have that  $Nu(V^{LB_{\rho_i}}) = PNu(V^{LB_{\rho_i}})$ .

**Main Result in Kar, Mitra and Wutuswami (2009):** If a TU-game  $V$  is a PS-game, then for all  $i \in N$ ,  $Sh_i(V) = PNu(V) = k_i/2$ , where  $PNu$  denotes the Prenucleolus and  $k_i$  is the player  $i$ 's specific constant corresponding to the sum of her marginal contribution to any pair of disjoint coalitions  $T, T^*$  such that  $T \cup T^* = N \setminus \{i\}$ . ■

## APPENDIX E

## PROOFS OF CHAPTER 6.

**Proof of Proposition 10.**

*Proposition 10: For each  $(E, c, P_1) \in B_P$ ,  $((E, c, P_2) \in B_P)$ ,  $((E, c, P_3) \in B_P)$ , the  $P$ -Safety for the smallest agent, say 1, corresponds with the Constrained Equal Losses (the Dual of Piniles') (the Dual Constrained Egalitarian) rule,*

$$s_1(E, c, P_1) = CEL_1(E, c) \quad (s_1(E, c, P_2) = DPin_1(E, c)) \quad (s_1(E, c, P_3) = DCE_1(E, c)).$$

- For each problem  $(E, c, P_1)$ , such that  $(E, c) \in \mathcal{B}_0$ , note that:

If  $CEL_1(E, c) = 0$ , by non-negativity,  $\varphi_1(E, c) = 0$ .

If  $CEL_1(E, c) > 0$ , by the definition of the  $CEL$  rule,  $c_1 - CEL_1(E, c) = c_j - CEL_j(E, c)$ , for each  $j \in N \neq 1$ .

So by efficiency, if  $\varphi_1(E, c) < CEL_1(E, c)$ ,  $c_1 - \varphi_1(E, c) > c_j - \varphi_j(E, c)$ , contradicting *Order Preservation*.

Therefore, there is no Admissible rule,  $\varphi$ , for  $P_1$  such that,  $\varphi_1(E, c) < CEL_1(E, c)$ . ■

- For each problem  $(E, c, P_2)$ , such that  $(E, c) \in \mathcal{B}_0$ , note that:

Case 1)  $E \leq C/2$

If  $DPin_1(E, c) = 0$ , by non-negativity,  $\varphi_1(E, c) = 0$ .

If  $DPin_1(E, c) > 0$ , by the definition of the  $DPin$  rule,  $\frac{c_1}{2} - DPin_1(E, c) = \frac{c_i}{2} - DPin_j(E, c)$ , for each  $j \in N \neq 1$ .

So by efficiency, if  $\varphi_1(E, c) < DPin_1(E, c)$ ,  $\frac{c_1}{2} - \varphi_1(E, c) > \frac{c_i}{2} - \varphi_j(E, c)$ . By the *Midpoint Property*, when  $E' = C/2$ ,  $\varphi(E', c) = c/2$ , so that,

$$\begin{aligned} \frac{c_1}{2} - \varphi_1(E, c) &> \frac{c_j}{2} - \varphi_j(E, c) \\ \varphi_1(E', c) - \varphi_1(E, c) &> \varphi_j(E', c) - \varphi_j(E, c), \end{aligned}$$

contradicting *Super-Modularity*.

Case 2)  $E \geq C/2$

If  $DPin_1(E, c) = c_1/2$ , by the *Midpoint Property*,  $\varphi_1(E, c) = c_1/2$ .

If  $DPin_1(E, c) > c_1/2$ , by the definition of the  $DPin$  rule,  $c_1 - DPin_1(E, c) = c_j - DPin_j(E, c)$ , for each  $j \in N \neq 1$ .

So by efficiency, if  $\varphi_1(E, c) < DPin_1(E, c)$ ,  $c_1 - \varphi_1(E, c) > c_j - \varphi_j(E, c)$ .

When  $E' = C$ ,  $\varphi(E', c) = c$ , so that,

$$\begin{aligned} c_1 - \varphi_1(E, c) &> c_j - \varphi_j(E, c) \\ \varphi_1(E', c) - \varphi_1(E, c) &> \varphi_j(E', c) - \varphi_j(E, c), \end{aligned}$$

contradicting *Super-Modularity*.

Therefore, there is no Admissible rule,  $\varphi$ , for  $P_2$  such that,  $\varphi_1(E, c) < DPin_1(E, c)$ . ■

- For each problem  $(E, c, P_3)$ , such that  $(E, c) \in \mathcal{B}_0$ , note that:

Case 1)  $E \leq C/2$

If  $DCE_1(E, c) = 0$ , by non-negativity,  $\varphi_1(E, c) = 0$ .

If  $DCE_1(E, c) > 0$ , by the definition of the *DCE* rule,  $\frac{a}{2} - DCE_1(E, c) \leq \leq \frac{c_j}{2} - DCE_j(E, c)$ , for each  $j \in N \neq 1$ .

Case 1.1) If  $DCE_j(E, c) = \frac{c_j}{2}$ , for each  $j \in N \neq 1$ .

By efficiency, if  $\varphi_1(E, c) < DCE_1(E, c)$ ,  $\varphi_j(E, c) > DCE_j(E, c) = \frac{c_j}{2}$ . By the *Midpoint Property*, when  $E' = C/2$ ,  $\varphi(E', c) = c/2$ , so that,

$$\varphi_j(E', c) = \frac{c_j}{2} < \varphi_j(E, c),$$

contradicting *Resource Monotonicity*.

Case 1.2) If  $DCE_j(E, c) \neq \frac{c_j}{2}$ , for each  $j \in N \neq 1$ .

By the definition of the *DCE* rule,  $\frac{a}{2} - DCE_1(E, c) = \frac{c_j}{2} - DCE_j(E, c)$ , for each  $j \in N \neq 1$ .

So by efficiency, if  $\varphi_1(E, c) < DCE_1(E, c)$ ,  $\frac{a}{2} - \varphi_1(E, c) > \frac{c_j}{2} - \varphi_j(E, c)$ , contradicting *Order Preservation*.

Case 1.3) If  $DCE_l(E, c) \neq \frac{c_l}{2}$ , and  $DCE_k(E, c) = \frac{c_k}{2}$  for each  $l, k \in N \neq 1$ .

We can reapplied the reasoning of Cases 1.1 and 1.2. In this sense, if

Case 1.3.1) If  $\varphi_l(E, c) > DCE_l(E, c)$ .

$\varphi$  contradicts *Order Preservation*.

Case 1.3.2) If  $\varphi_k(E, c) > DCE_k(E, c) = \frac{c_k}{2}$ .

By the *Midpoint Property*,  $\varphi$  contradicts *Resource Monotonicity*.

Case 2)  $E \geq C/2$

If  $DCE_1(E, c) = c_1/2$ , by the *Midpoint Property*,  $\varphi_1(E, c) = c_1/2$ .

If  $DCE_1(E, c) > c_1/2$ , by the definition of the *DCE* rule,  $c_1 - DCE_1(E, c) = c_j - DCE_j(E, c)$ , for each  $j \in N \neq 1$ .

So by efficiency, if  $\varphi_1(E, c) < DCE_1(E, c)$ ,  $c_1 - \varphi_1(E, c) > c_j - \varphi_j(E, c)$ , contradicting *Order Preservation*.

Therefore, there is no Admissible rule,  $\varphi$ , for  $P_3$  such that,  $\varphi_1(E, c) < DCE_1(E, c)$ . ■

**Example 4** Let us consider the problem  $(E, c, P_1)$ , with  $(E, c) = (49, (18; 27; 40)) \in \mathcal{B}_P$ . Thus, we obtain that,  $CEA(E, c) = (16.3; 16.3; 16.3)$  and  $CEL(E, c) = (6; 15; 28)$ . By Proposition 10,  $s_1(E, c, P_1) = 6$  and  $s_3(E, c, P_1) = 16.3$ . However, for agent 2 neither of both rules is the smallest amount she can get according to  $P_1$ , since, for example, computing the Talmud rule,  $T(E, c) = (9; 13.5; 26.5)$ , we can see that  $T_2(E, c) = 13.5 < CEL_2(E, c) < CEA_2(E, c)$ . ■

**Example 5** Let us consider the problem  $(E, c) = (60, (15; 16; 30)) \in \mathcal{B}$ . On the one hand, by Proposition 10, for each  $(E, c, P_1)$  in  $\mathcal{B}_P$  and each  $m \in N$  the *P-Safety* for the smallest and the highest agents are the *CEL* and the *CEA* rules. On the other hand, it can be easily checked, following the same reasoning as the proof of Proposition 10, that the Admissible rule which recommends the smallest amount for the intermediate agent,  $\varphi^{P_1 \min 2}$ , is defined as follows,

Case 1)  $CEL_2(E, c) \leq CEA_2(E, c)$ .

$$\varphi^{P_1 \min 2}(E, c) = CEL(E, c) + (2\alpha; -\alpha; -\alpha), \text{ where}$$

$$\alpha = \begin{cases} \frac{CEL_2(E, c) - CEL_1(E, c)}{3} & \text{If } \frac{2}{3}(CEL_2(E, c) - CEL_1(E, c)) \leq c_1 \\ \frac{c_1 - CEL_1(E, c)}{2} & \text{Otherwise} \end{cases}$$

Case 2)  $CEL_2(E, c) \geq CEA_2(E, c)$ .

$$a) CEA_1(E, c) = CEA_2(E, c) = CEA_3(E, c).$$

$$a.1) c_3 - c_2 \leq E.$$

$$\varphi^{P_1 \min 2}(E, c) = CEL(E, c) + (-\alpha; -\alpha; 2\alpha), \text{ where}$$

$$\alpha = \frac{c_3 - c_2}{3}.$$

$$a.2) c_3 - c_2 \geq E.$$

$$\varphi^{P_1 \min 2}(E, c) \equiv CEL(E, c) = (0; 0; E).$$

$$b) CEA_1(E, c) \neq CEA_2(E, c) = CEA_3(E, c).$$

$$b.1) c_3 - c_2 \geq CEA_2(E, c) - CEA_1(E, c).$$

$$\varphi^{P_1 \min 2}(E, c) = CEA(E, c) + (-\alpha; -\alpha - \alpha^*; 2\alpha + \alpha^*), \text{ where}$$

$$\alpha^* = CEA_2(E, c) - CEA_1(E, c), \text{ and}$$

$$\alpha = \frac{c_3 - c_2 - 2\alpha^*}{3}.$$

$$b.2) c_3 - c_2 \leq CEA_2(E, c) - CEA_1(E, c).$$

$$\varphi^{P_1 \min 2}(E, c) = CEA(E, c) + (0; -\alpha; \alpha), \text{ where}$$

$$\alpha = \frac{c_3 - c_2}{2}.$$

Note that the case  $CEA_1(E, c) \neq CEA_2(E, c) \neq CEA_3(E, c)$  is not possible, since, then by the definition of the Constrained Equal Awards and the Constrained Equal Losses rules,  $CEL_2(E, c) < CEA_2(E, c)$ .

Moreover, by applying the idea of duality,  $\varphi^{P_1 \max 2}$  denotes the Admissible rule which recommends the highest amount for the intermediate agent. That is,

$$\varphi^{P_1 \max 2}(E, c) = c_i - \varphi^{P_1 \min 2}(L, c).$$

Then,

$$\text{Step } m = 1, (E^1, c^1) = (60, (15; 16; 30)),$$

$$CEA(E^1, c^1) = (15; 16; 29),$$

$$CEL(E^1, c^1) = (14.66; 15.66; 29.66),$$

$$\varphi^{P_1 \min 2}(E^1, c^1) = (15; 15.5; 29.5),$$

$$\varphi^{P_1 \max^2}(E^1, c^1) = (15; 16; 29).$$

Then,

$$s(E^1, c^1, P_1) = (14.66; 15.5; 29),$$

$$ce(E^1, c^1, P_1) = (15; 16; 29.66).$$

$$\text{Step } m = 2 : (E^2, c^2) = (0.833, (0.33; 0.5; 0.66)),$$

$$CEA(E^2, c^2) = 0.278; 0.278; 0.278,$$

$$CEL(E^2, c^2) = (0.111; 0.278; 0.444),$$

$$\varphi^{P_1 \min^2}(E^2, c^2) = (0.222; 0.222; 0.389),$$

$$\varphi^{P_1 \max^2}(E^2, c^2) = (0.167; 0.333; 0.333).$$

Then,

$$s(E^2, c^2, P_1) = (0.111; 0.222; 0.278),$$

$$ce(E^2, c^2, P_1) = (0.278; 0.333; 0.444).$$

$$\text{Step } m = 3 : (E^3, c^3) = (0.222, (0.166; 0.111; 0.166)),$$

$$CEA(E^3, c^3) = (0.074; 0.074; 0.074),$$

$$CEL(E^3, c^3) = (0.092; 0.037; 0.092),$$

Note that in this step, the smallest agent is agent 2, and agent 1

and 3 has the same claim, so

$$s(E^3, c^3, P_1) = (0.074; 0.037; 0.074),$$

$$ce(E^3, c^3, P_1) = (0.092; 0.074; 0.092).$$

At this point,

$$\sum_{k=1}^3 s(E^k, c^k, P_1) = (14.851; 15.759; 29.185).$$

By the definition of the *P*-Safety (Definition 18),  $s_i(E^m, c^m, P_1) \geq 0$ , therefore,

$$\begin{aligned} DBR(E, c, P_3) &= (14.851; 15.759; 29.185) + \sum_{m=4}^{\infty} s(E^m, c^m, P_1) \neq \\ &\neq \frac{ce(E, c, P_1) + s(E, c, P_1)}{2} = (14.83; 15.75; 29.33). \end{aligned}$$

Moreover, the average of the *P*-Safety and the *P*-Ceiling does not exhaust the resource,

$$\frac{ce(E,c,P_i)+s(E,c,P_i)}{2} = (14.83; 15.75; 29.33). \blacksquare$$

**Example 6** Let us consider the bankruptcy problem  $\mathcal{B}$ , in which  $E = 60$  and there are three focal distribution rules,  $f$ ,  $g$  and  $h$ , providing  $f(E, c) = (10, 25, 25)$ ,  $g(E, c) = (0, 27.5, 32.5)$  and  $h(E, c) = (10, 22.5, 27.5)$ . If the associated TU-game is defined by

$$V^{B_{P_i}}(S) = \min \left\{ \sum_{i \in S} f_i(E, c), \sum_{i \in S} g_i(E, c), \sum_{i \in S} h_i(E, c) \right\},$$

it is easy to verify that  $V^{B_{P_i}}(\{1\}) = 0$ ,  $V^{B_{P_i}}(\{2\}) = 22.5$ ,  $V^{B_{P_i}}(\{3\}) = 25$ ,  $V^{B_{P_i}}(\{1, 2\}) = 27.5$ ,  $V^{B_{P_i}}(\{1, 3\}) = 32.5$ ,  $V^{B_{P_i}}(\{2, 3\}) = 50$  and  $V^{B_{P_i}}(\{1, 2, 3\}) = 60$ . Then  $Sh(V^{B_{P_i}}) = (5.42, 25.42, 29.16)$ ,  $Nu(V^{B_{P_i}}) = (6.25, 25, 28.75)$  and  $(f + g + h)/3 = (6 + (2/3), 25, 28 + (1/3))$ .

Note that in this example the three focal distribution rules are involved in specifying the coalitions worth and this fact causes the loss of the PS-game quality. Furthermore, even  $h$  only determines the worth of player 2, none of the players  $i, i \in \{1, 2, 3\}$ , has a constant sum of his marginal contribution to all pairs of coalitions whose union is  $N \setminus \{i\}$ :

$$\begin{aligned} \Delta_1 V(\{2\}) + \Delta_1 V(\{3\}) &= 12.5 \neq 10 = \Delta_1 V(\{\emptyset\}) + \Delta_1 V(\{2, 3\}), \\ \Delta_2 V(\{1\}) + \Delta_2 V(\{3\}) &= 52.5 \neq 50 = \Delta_2 V(\{\emptyset\}) + \Delta_2 V(\{1, 3\}) \text{ and} \\ \Delta_3 V(\{1\}) + \Delta_3 V(\{2\}) &= 60 \neq 57.5 = \Delta_3 V(\{\emptyset\}) + \Delta_3 V(\{1, 2\}). \blacksquare \end{aligned}$$



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